

Rational Surface Automorphisms with Positive Entropy ^{*}

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Abstract

The aim of this paper is to construct many examples of rational surface automorphisms with positive entropy by means of the concept of orbit data. We show that if an orbit data satisfies some mild conditions, then there exists an automorphism realizing the orbit data. Applying this result, we describe the set of entropy values of the rational surface automorphisms in terms of Weyl groups.

1 Introduction

In this paper, we consider automorphisms on compact complex surfaces with positive entropy. According to a result of S. Cantat [4], a surface admitting an automorphism with positive entropy must be either a K3 surface, an Enriques surface, a complex torus or a rational surface. For rational surfaces, rather few examples had been known (see [4], Section 2). However, some rational surface automorphisms with invariant cuspidal anticanonical curves have been constructed recently. Bedford and Kim [2, 3] found some examples of automorphisms by studying an explicit family of quadratic birational maps on \mathbb{P}^2 , and then McMullen [10] gave a synthetic construction of many examples. More recently, Diller [5] sought automorphisms from quadratic maps that preserve a cubic curve by using the group law for the cubic curve. We stress the point that these automorphisms can be all obtained from quadratic birational maps. The aim of this paper is to construct yet more examples of rational surface automorphisms with positive entropy from general birational maps on \mathbb{P}^2 preserving a cuspidal cubic curve.

Let $F : X \rightarrow X$ be an automorphism on a rational surface X . From results of Gromov and Yomdin [7, 13], the *topological entropy* $h_{\text{top}}(F)$ of F is calculated as $h_{\text{top}}(F) = \log \lambda(F^*)$, where $\lambda(F^*)$ is the spectral radius of the action $F^* : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ on the cohomology group. Therefore, when handling the topological entropy of a map, we need to discuss its action on the

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cohomology group, which can be described as an element of a Weyl group acting on a Lorentz lattice. The *Lorentz lattice* $\mathbb{Z}^{1,N}$ is the lattice with the Lorentz inner product given by

$$\mathbb{Z}^{1,N} = \bigoplus_{i=0}^N \mathbb{Z} \cdot e_i, \quad (e_i, e_j) = \begin{cases} 1 & (i = j = 0), \\ -1 & (i = j = 1, \dots, N), \\ 0 & (i \neq j). \end{cases} \quad (1)$$

For $N \geq 3$, the Weyl group $W_N \subset O(\mathbb{Z}^{1,N})$ is the group generated by $(\rho_i)_{i=0}^{N-1}$, where $\rho_i : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$ is the reflection defined by

$$\rho_i(x) = x + (x, \alpha_i) \cdot \alpha_i, \quad \alpha_i := \begin{cases} e_0 - e_1 - e_2 - e_3 & (i = 0), \\ e_i - e_{i+1} & (i = 1, \dots, N-1). \end{cases} \quad (2)$$

We call the W_N -translate $\Phi_N := \bigcup_{i=0}^{N-1} W_N \cdot \alpha_i$ of the elements $(\alpha_0, \dots, \alpha_{N-1})$ the *root system* of W_N , and each element of Φ_N a *root*. On the other hand, if $\lambda(F^*) > 1$, then there is a blowup $\pi : X \rightarrow \mathbb{P}^2$ of N points (p_1, \dots, p_N) (see [11]), which gives an expression of the cohomology group : $H^2(X; \mathbb{Z}) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \dots \oplus \mathbb{Z}[E_N]$, where H is the total transform of a line in \mathbb{P}^2 , and E_i is the total transform over p_i . Moreover, there is a natural marking isomorphism $\phi_\pi : \mathbb{Z}^{1,N} \rightarrow H^2(X, \mathbb{Z})$, sending the basis as $\phi_\pi(e_0) = [H]$ and $\phi_\pi(e_i) = [E_i]$ for $i = 1, \dots, N$. It is known (see [12]) that there is a unique element $w \in W_N$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^{1,N} & \xrightarrow{w} & \mathbb{Z}^{1,N} \\ \phi_\pi \downarrow & & \downarrow \phi_\pi \\ H^2(X, \mathbb{Z}) & \xrightarrow{F^*} & H^2(X; \mathbb{Z}). \end{array} \quad (3)$$

Then w is said to be *realized* by (π, F) (see also [10]). A question at this stage is whether a given element $w \in W_N$ is realized by some pair (π, F) .

Again let us consider a blowup $\pi : X \rightarrow \mathbb{P}^2$ and an automorphism $F : X \rightarrow X$. Through π , F descends to a birational map $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ on the projective plane \mathbb{P}^2 , and it, in turn, is expressed as a composition $f = f_n \circ f_{n-1} \circ \dots \circ f_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of quadratic birational maps $f_i : \mathbb{P}_{i-1}^2 \rightarrow \mathbb{P}_i^2$ with $\mathbb{P}_i^2 = \mathbb{P}^2$ from Noether's theorem. Since the inverse of any quadratic map is also a quadratic map and a quadratic map has three points of indeterminacy, we denote the indeterminacy sets of f_i and of f_i^{-1} by $I(f_i) = \{p_{i,1}^+, p_{i,2}^+, p_{i,3}^+\} \subset \mathbb{P}_{i-1}^2$ and $I(f_i^{-1}) = \{p_{i,1}^-, p_{i,2}^-, p_{i,3}^-\} \subset \mathbb{P}_i^2$ respectively. Write $p_{\bar{\iota}}^\pm = p_{i,\iota}^\pm$ with $\bar{\iota} \in \mathcal{K}(n) := \{\bar{\iota} = (i, \iota) \mid i = 1, 2, \dots, n, \iota = 1, 2, 3\}$. Then, there is a unique permutation σ of $\mathcal{K}(n)$ and a unique function $\mu : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ such that the following condition holds for any $\bar{\iota} \in \mathcal{K}(n)$:

$$p_{\bar{\iota}}^m \neq p_{\bar{\iota}'}^+ \quad (0 \leq m < \mu(\bar{\iota}), \bar{\iota}' \in \mathcal{K}(n)), \quad p_{\bar{\iota}}^{\mu(\bar{\iota})} = p_{\sigma(\bar{\iota})}^+, \quad (4)$$

where $p_{\bar{\iota}}^0 := p_{\bar{\iota}}^-$, and for $m \geq 1$, $p_{\bar{\iota}}^m$ is defined inductively by $p_{\bar{\iota}}^m := f_r(p_{\bar{\iota}}^{m-1}) \in \mathbb{P}_r^2$ with $r \equiv i + m \pmod{n}$. Moreover, we denote by $\kappa(\bar{\iota})$ the number of points among $p_{\bar{\iota}}^0, p_{\bar{\iota}}^1, \dots, p_{\bar{\iota}}^{\mu(\bar{\iota})}$ lying on \mathbb{P}_n^2 or, in other words, $\kappa(\bar{\iota}) = (\mu(\bar{\iota}) + i + 1 - i_1)/n$ with $(i_1, \iota_1) := \sigma(\bar{\iota})$. It is easy to see that $\kappa(\bar{\iota}) \geq 1$ provided $i_1 \leq i$. This observation leads us to the following definition.

Definition 1.1 An *orbit data* is a triplet $\tau = (n, \sigma, \kappa)$ consisting of

- a positive integer n ,
- a permutation σ of $\mathcal{K}(n) := \{(i, \iota) \mid i = 1, 2, \dots, n, \iota = 1, 2, 3\}$, and
- a function $\kappa : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\kappa(\bar{\iota}) \geq 1$ provided $i_1 \leq i$, where $(i_1, \iota_1) = \sigma(\bar{\iota})$.

Note that an orbit data τ restores the function $\mu : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ given by $\mu(\bar{\iota}) = \kappa(\bar{\iota}) \cdot n + i_1 - i - 1$.

Definition 1.2 An n -tuple $\bar{f} = (f_1, \dots, f_n)$ of quadratic birational maps f_i is called a *realization* of an orbit data τ if condition (4) holds for any $\bar{\iota} \in \mathcal{K}(n)$.

A question here is whether a given orbit data τ admits some realization \bar{f} .

To answer this, we consider a class of birational maps preserving a cuspidal cubic C on \mathbb{P}^2 . Let $\mathcal{Q}(C)$ be the set of quadratic birational maps $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ satisfying $f(C) = C$ and $I(f) \subset C^*$, where C^* is the smooth locus of C . The smooth locus C^* is isomorphic to \mathbb{C} and is preserved by any map $f \in \mathcal{Q}(C)$. Thus, the restriction $f|_{C^*}$ is an automorphism expressed as

$$f|_{C^*} : C^* \rightarrow C^*, \quad t \mapsto \delta(f) \cdot t + k_f$$

for some $\delta(f) \in \mathbb{C}^\times$ and $k_f \in \mathbb{C}$. For an n -tuple $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$, we define the *determinant* of \bar{f} by $\delta(\bar{f}) := \prod_{i=1}^n \delta(f_i)$. Moreover, to state our main theorems, we introduce the condition

$$w_\tau^{\ell_\tau}(\alpha) \neq \alpha \quad (\alpha \in \Gamma_\tau), \quad (5)$$

where w_τ is an element of W_N with $N := \sum_{\bar{\iota} \in \mathcal{K}(n)} \kappa(\bar{\iota})$, ℓ_τ is a positive integer and Γ_τ is a *finite* subset of Φ_N , which are canonically determined by τ . These definitions will be given in Section 2 (see Definitions 2.1, 2.2 and 2.4). Condition (5) is referred to as the *realizability condition*, for reasons that become clear in the following theorem.

Theorem 1.3 Assume that an orbit data τ satisfies $\lambda(w_\tau) > 1$ and the realizability condition (5). Then, there is a unique realization $\bar{f}_\tau = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ of τ such that $\delta(\bar{f}_\tau) = \lambda(w_\tau)$. Moreover, τ determines a blowup $\pi_\tau : X_\tau \rightarrow \mathbb{P}^2$ of N points on C^* in a canonical way, which lifts $f_\tau := f_n \circ \dots \circ f_1$ to an automorphism $F_\tau : X_\tau \rightarrow X_\tau$:

$$\begin{array}{ccc} X_\tau & \xrightarrow{F_\tau} & X_\tau \\ \pi_\tau \downarrow & & \downarrow \pi_\tau \\ \mathbb{P}^2 & \xrightarrow{f_\tau} & \mathbb{P}^2. \end{array}$$

Finally, (π_τ, F_τ) realizes w_τ and F_τ has positive entropy $h_{\text{top}}(F_\tau) = \log \lambda(w_\tau) > 0$.

As seen in Theorem 1.5, almost all orbit data satisfy the realizability condition (5). Furthermore, even if an orbit data τ does not satisfy the realizability condition (5), its sibling $\check{\tau}$ does satisfy the condition.

Theorem 1.4 *For any orbit data τ with $\lambda(w_\tau) > 1$, there is an orbit data $\tilde{\tau}$ satisfying $\lambda(w_{\tilde{\tau}}) = \lambda(w_\tau) > 1$ and the realizability condition (5), and thus $\tilde{\tau}$ is realized by $\bar{f}_{\tilde{\tau}}$.*

Moreover, we give a sufficient condition for (5), which enables us to see clearly that almost all orbit data are realized, and to obtain an estimate for the entropy.

Theorem 1.5 *Assume that an orbit data $\tau = (n, \sigma, \kappa)$ satisfies*

- (1) $n \geq 2$,
- (2) $\kappa(\bar{\tau}) \geq 3$ for any $\bar{\tau} \in \mathcal{K}(n)$, and
- (3) if $\bar{\tau} \neq \bar{\tau}'$ satisfy $i_m = i'_m$ and $\kappa(\bar{\tau}_m) = \kappa(\bar{\tau}'_m)$ for any $m \geq 0$, then $\bar{\tau}_m \neq \bar{\tau}'_m$ for any $m \geq 0$, where $\bar{\tau}_m = (i_m, \iota_m) := \sigma^m(\bar{\tau})$.

Then the orbit data τ satisfies $2^n - 1 < \lambda(w_\tau) < 2^n$ and the realizability condition (5). In particular, F_τ has positive entropy $\log(2^n - 1) < h_{\text{top}}(F_\tau) < \log 2^n$.

Diller [5] constructs, by studying single quadratic maps preserving C , automorphisms with positive entropy realizing orbit data $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$. As is seen in Example 2.5, there is an orbit data τ such that F_τ is not topologically conjugate to the iterates of $F_{\hat{\tau}}$ that Diller constructs for any $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$.

Now, let us come back to consider an element w of the Weyl group W_N . It is connected with orbit data by the fact that w is expressed as $w = w_\tau$ for some orbit data τ (see Proposition 2.6). Thus, Theorem 1.3 extends the result of McMullen [10] which states that if w has spectral radius $\lambda(w) > 1$ and no periodic roots in Φ_N , that is, $w^k(\alpha) \neq \alpha$ for any $\alpha \in \Phi_N$ and $k \geq 1$, then w is realized by a pair (π, F) . However, since the roots and the periods are infinite, it is rather difficult to see whether w has no periodic roots. On the other hand, in condition (5), the set Γ_τ is finite and the period ℓ_τ is fixed. Thus, once an orbit data τ with $w = w_\tau$ is fixed, it is easier to check that w satisfies condition (5). In Example 2.5, we give an example of w realized by a pair (π, F) and admitting periodic roots.

In general, the topological entropy of any automorphism $F : X \rightarrow X$ is expressed as $h_{\text{top}}(F) = \log \lambda(w)$ for some $w \in W_N$ (see Proposition 4.3). Conversely, by Theorem 1.5, the logarithm of an arbitrary element in the set

$$\Lambda := \{\lambda(w) \geq 1 \mid w \in W_N, N \geq 3\} \quad (6)$$

gives rise to the entropy of some automorphism.

Corollary 1.6 *For any $\lambda \in \Lambda$, there is an automorphism $F : X \rightarrow X$ of a rational surface X such that $h_{\text{top}}(F) = \log \lambda$. Moreover, we have*

$$\{h_{\text{top}}(F) \mid F : X \rightarrow X \text{ is a rational surface automorphism}\} = \{\log \lambda \mid \lambda \in \Lambda\}.$$

The core of the proofs of these theorems is to find concretely the configuration of the points $\{p_\tau^0, \dots, p_\tau^{\mu(\tau)}\}_{\tau \in \mathcal{K}(n)}$, which are blown up to yield an automorphism. Indeed, the configuration is determined by an eigenvector of w_τ . Then, our investigations on the existence of a realization

are divided into two steps. The first step is to check that τ admits a tentative realization (see Definition 5.1). Tentative realization is a necessary condition for realization. Moreover, Proposition 5.7 states that a tentative realization \bar{f} of τ with $\delta(\bar{f}) = \lambda(w_\tau)$ exists if and only if w_τ has no periodic roots in a finite subset $\Gamma_\tau^{(1)}$ of Γ_τ . The second step is to check that τ is compatible with the configuration as in Proposition 6.2, or that the tentative realization \bar{f} is indeed a realization of τ . Proposition 6.3 shows that \bar{f} is a realization of τ if and only if w_τ has no periodic roots in $\Gamma_\tau^{(2)} := \Gamma_\tau \setminus \Gamma_\tau^{(1)}$. When τ does not pass these two inspections, its sibling $\tilde{\tau}$ with $\lambda(w_{\tilde{\tau}}) = \lambda(w_\tau)$, determining essentially the same configuration as τ , satisfies the realizability condition (5) and admits a realization. On the other hand, under the assumptions in Theorem 1.5, Proposition 5.9 gives an estimate for the spectral radius $\lambda(w_\tau)$ and shows the absence of periodic roots in $\Gamma_\tau^{(1)}$, and then Proposition 6.5 guarantees the absence of periodic roots in $\Gamma_\tau^{(2)}$, which proves Theorem 1.5.

This article is organized as follows. After defining the element $w_\tau \in W_N$, the integer ℓ_τ and the finite subset Γ_τ in Section 2, we describe eigenvectors of w_τ explicitly in Section 3. Section 4 is devoted to giving a method for constructing a rational surface automorphism from a realization of τ . In Section 5, we discuss the existence of a tentative realization of τ , and in Section 6, we investigate whether it is indeed a realization and prove Theorems 1.3–1.5 and Corollary 1.6. Finally, Propositions 5.9 and 6.5 are proved in Section 7.

2 Definitions and Example

As is mentioned in the Introduction, an orbit data τ canonically determines the element $w_\tau \in W_N$, the integer ℓ_τ , and the finite subset Γ_τ of Φ_N , which appear in the realizability condition (5). In this section, we give these definitions, and also give an example of an orbit data that admits a realization. Moreover, it is shown in the last part of this section (see Proposition 2.6) that any element w in W_N is expressed as $w = w_\tau$ for some orbit data τ .

First, we recall the Weyl group action on $\mathbb{Z}^{1,N}$ for $N \geq 3$, where $\mathbb{Z}^{1,N}$ is the Lorentz lattice with Lorentz inner product given in (1). The Weyl group $W_N \subset O(\mathbb{Z}^{1,N})$ is the group generated by the reflections $(\rho_i)_{i=0}^{N-1}$ given in (2), which preserves the Lorentz inner product on $\mathbb{Z}^{1,N}$. The W_N -translate $\Phi_N := \bigcup_{i=0}^{N-1} W_N \cdot \alpha_i$ of the elements (α_i) is called the *root system* of W_N , and each element of Φ_N is called a *root* of W_N .

For an orbit data $\tau = (n, \sigma, \kappa)$ (see Definition 1.1), we consider the lattice

$$L_\tau := \mathbb{Z}e_0 \oplus \left(\bigoplus_{\bar{\tau} \in \mathcal{K}(n)} \bigoplus_{k=1}^{\kappa(\bar{\tau})} \mathbb{Z}e_\tau^k \right) \cong \mathbb{Z}^{1,N} \quad \left(N = \sum_{\bar{\tau} \in \mathcal{K}(n)} \kappa(\bar{\tau}) \right),$$

with the inner product given by

$$\begin{cases} (e_0, e_0) = 1 \\ (e_\tau^k, e_\tau^k) = -1 & (\bar{\tau} \in \mathcal{K}(n), \quad 1 \leq k \leq \kappa(\bar{\tau})) \\ (e_0, e_\tau^k) = (e_\tau^k, e_{\tau'}^{k'}) = 0 & ((\bar{\tau}, k) \neq (\bar{\tau}', k')). \end{cases}$$

Then the automorphism $r_\tau : L_\tau \rightarrow L_\tau$ is defined by

$$r_\tau : \begin{cases} e_0 & \mapsto e_0 \\ e_{\sigma_\tau(\bar{\iota})}^1 & \mapsto e_{\bar{\iota}}^{\kappa(\bar{\iota})} \\ e_{\bar{\iota}}^k & \mapsto e_{\bar{\iota}}^{k-1} \quad (2 \leq k \leq \kappa(\bar{\iota})), \end{cases}$$

where $\sigma_\tau(\bar{\iota}) = \sigma^m(\bar{\iota})$ with $m \geq 1$ determined by the relations $\kappa(\sigma^k(\bar{\iota})) = 0$ for $1 \leq k < m$, and $\kappa(\sigma^m(\bar{\iota})) \geq 1$. Note that σ_τ becomes a permutation of $\{\bar{\iota} \in \mathcal{K}(n) \mid \kappa(\bar{\iota}) \geq 1\}$, and so $e_{\sigma_\tau(\bar{\iota})}^1$ is well-defined. The automorphism r_τ is an element of the subgroup $\langle \rho_1, \dots, \rho_{N-1} \rangle \subset W_N$ generated by $\rho_1, \dots, \rho_{N-1}$. On the other hand, for $1 \leq j \leq n$, the automorphism $q_j : L_\tau \rightarrow L_\tau$ is defined by

$$q_j : \begin{cases} e_0 & \mapsto 2e_0 - \sum_{\iota=1}^3 e_{(j,\iota)_\tau}^1 \\ e_{(j,\iota^{(1)})_\tau}^1 & \mapsto e_0 - e_{(j,\iota^{(2)})_\tau}^1 - e_{(j,\iota^{(3)})_\tau}^1 \quad (\{\iota^{(1)}, \iota^{(2)}, \iota^{(3)}\} = \{1, 2, 3\}) \\ e_{\bar{\iota}}^k & \mapsto e_{\bar{\iota}}^k \quad (\text{otherwise}), \end{cases}$$

where $(j, \iota^{(\nu)})_\tau = \sigma^{m_\nu}(j, \iota^{(\nu)})$ with $m_\nu \geq 0$ determined by the relations $\kappa(\sigma^k(j, \iota^{(\nu)})) = 0$ for $0 \leq k < m_\nu$, and $\kappa(\sigma^{m_\nu}(j, \iota^{(\nu)})) \geq 1$. We notice that if $\iota^{(1)} \neq \iota^{(2)}$ then $(j, \iota^{(1)})_\tau \neq (j, \iota^{(2)})_\tau$. Indeed, assume the contrary that $\sigma^{m_1}(j, \iota^{(1)}) = \sigma^{m_2}(j, \iota^{(2)})$ for $m_1 > m_2$, or $\sigma^m(j, \iota^{(1)}) = (j, \iota^{(2)})$ for $m = m_1 - m_2 > 0$. As $(j_k, (\iota^{(1)})_k) := \sigma^k(j, \iota^{(1)})$ satisfies $\kappa(j_k, (\iota^{(1)})_k) = 0$ for $0 \leq k \leq m-1 \leq m_1 - 1$, one has $j = j_0 < j_1 < \dots < j_m = j$, which is a contradiction. The automorphism q_j is conjugate to ρ_0 under the action of $\langle \rho_1, \dots, \rho_{N-1} \rangle$.

Now we define the lattice automorphism $w_\tau : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$.

Definition 2.1 For an orbit data $\tau = (n, \sigma, \kappa)$, we define the lattice automorphism $w_\tau : L_\tau \rightarrow L_\tau$ by

$$w_\tau := r_\tau \circ q_1 \circ \dots \circ q_n : L_\tau \rightarrow L_\tau.$$

We sometimes write $w_\tau : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$.

Indeed, it is easily seen that w_τ is an element of W_N . Through the isomorphism $w_\tau : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$, the definition of the integer ℓ_τ is given in the following manner.

Definition 2.2 The positive integer ℓ_τ is defined as the minimal positive integer satisfying $d^{\ell_\tau} = 1$ for any eigenvalue d of w_τ that is a root of unity.

Before determining the finite set Γ_τ , we define a set $\mathcal{T}(\tau)$ of n -tuples $(\prec_i)_{i=1}^n$ of total orders \prec_i on the subset $\mathcal{K}(n)_i := \{(i, 1), (i, 2), (i, 3)\}$ of $\mathcal{K}(n)$ (see also Remark 5.10). Recall that the orbit data $\tau = (n, \sigma, \kappa)$ defines the function $\mu : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\mu(\bar{\iota}) := \kappa(\bar{\iota}) \cdot n + i_1 - i - 1 = \theta_{i, i_1-1}(\kappa(\bar{\iota})), \quad (7)$$

where $\bar{\iota}_m = (i_m, \iota_m) = \sigma^m(\bar{\iota})$ for $\bar{\iota} = (i, \iota) \in \mathcal{K}(n)$, and

$$\theta_{i, i'}(k) := k \cdot n + i' - i. \quad (8)$$

Moreover, let $G_\tau : \{1, \dots, n\} \rightarrow \{1, 2, 3\}$ be the function defined by

$$G_\tau(i) := \begin{cases} 1 & (\mu(\bar{t}) \neq \mu(\bar{t}') \text{ for } \bar{t} \neq \bar{t}' \in \mathcal{K}(n)_i) \\ 2 & (\mu(\bar{t}) = \mu(\bar{t}') \neq \mu(\bar{t}'') \text{ for } \{\bar{t}, \bar{t}', \bar{t}''\} = \mathcal{K}(n)_i) \\ 3 & (\mu(\bar{t}) = \mu(\bar{t}') = \mu(\bar{t}'') \text{ for } \{\bar{t}, \bar{t}', \bar{t}''\} = \mathcal{K}(n)_i). \end{cases}$$

Definition 2.3 Let $\mathcal{T}(\tau)$ be the set of all n -tuples $(\prec_i)_{i=1}^n$ of total orders \prec_i on $\mathcal{K}(n)_i$ inductively satisfying the following conditions. First, for the minimal integer $i^{(1)} \in \mathcal{N}(0) := \{1, \dots, n\}$ such that $G_\tau(i^{(1)}) = \min\{G_\tau(i) \mid i \in \mathcal{N}(0)\}$, the total order $\prec_{i^{(1)}}$ satisfies $\bar{t} \prec_{i^{(1)}} \bar{t}'$ provided $\mu(\bar{t}) < \mu(\bar{t}')$. Moreover, for $1 \leq m \leq n-1$, suppose that $i^{(m)} \in \mathcal{N}(m-1)$ is given. Put $\mathcal{N}(m) := \mathcal{N}(m-1) \setminus \{i^{(m)}\}$. If there are distinct elements $\bar{t}, \bar{t}' \in \mathcal{K}(n)_{i^{(m+1)}}$ for some $i^{(m+1)} \in \mathcal{N}(m)$ such that $i^{(m)} = i_1 = i'_1$ and $\mu(\bar{t}) = \mu(\bar{t}')$, then $\prec_{i^{(m+1)}}$ satisfies $\bar{t} \prec_{i^{(m+1)}} \bar{t}'$ provided that either $\mu(\bar{t}) = \mu(\bar{t}')$ and $\bar{t}_1 \prec_{i^{(m)}} \bar{t}'_1$, or $\mu(\bar{t}) < \mu(\bar{t}')$. Otherwise, for the minimal integer $i^{(m+1)} \in \mathcal{N}(m)$ such that $G_\tau(i^{(m+1)}) = \min\{G_\tau(i) \mid i \in \mathcal{N}(m)\}$, the total order $\prec_{i^{(m+1)}}$ satisfies $\bar{t} \prec_{i^{(m+1)}} \bar{t}'$ provided $\mu(\bar{t}) < \mu(\bar{t}')$.

We define the finite subsets of the root system by

$$\Gamma_\tau^{(1)} := \{\alpha_j^c \mid j = 1, \dots, n\} \subset \Phi_N, \quad (9)$$

$$\Gamma_\tau^{(2)} := \bar{\Gamma}_\tau^{(2)} \setminus \check{\Gamma}_\tau^{(2)} \subset \Phi_N, \quad (10)$$

$$\bar{\Gamma}_\tau^{(2)} := \{\alpha_{\bar{t}, \bar{t}'}^k \mid \bar{t} = (i, \iota), \bar{t}' = (i', \iota') \in \mathcal{K}(n), 0 \leq \theta_{i, i'}(k) \leq \mu(\bar{t})\} \subset \Phi_N,$$

where α_j^c and $\alpha_{\bar{t}, \bar{t}'}^k$ are the roots given by

$$\alpha_j^c := q_n \circ \dots \circ q_{j+1}(e_0 - e_{(j,1)\tau}^1 - e_{(j,2)\tau}^1 - e_{(j,3)\tau}^1), \quad (11)$$

$$\alpha_{\bar{t}, \bar{t}'}^k := q_n \circ \dots \circ q_{i'+1}(e_{\bar{t}}^{k+1} - e_{\bar{t}'}^1). \quad (12)$$

Moreover, $\check{\Gamma}_\tau^{(2)}$ is the set of roots $\alpha_{\bar{t}, \bar{t}'}^k$ in $\bar{\Gamma}_\tau^{(2)}$ satisfying the following conditions for a given $(\prec_i) \in \mathcal{T}(\tau)$:

- (1) If $\theta_{i, i'}(k) > 0$, then either $\mu(\bar{t}) = \mu(\bar{t}') + \theta_{i, i'}(k)$ and $\bar{t}'_1 \prec_{i_1} \bar{t}_1$, or $\mu(\bar{t}) > \mu(\bar{t}') + \theta_{i, i'}(k)$.
- (2) When $\theta_{i, i'}(k) = 0$, then $\bar{t}' \prec_i \bar{t}$ if and only if either $\mu(\bar{t}) = \mu(\bar{t}')$ and $\bar{t}'_1 \prec_{i_1} \bar{t}_1$, or $\mu(\bar{t}) > \mu(\bar{t}')$.

Indeed, the definitions of $\check{\Gamma}_\tau^{(1)}$ and $\Gamma_\tau^{(1)}$ are independent of the choice of $(\prec_i) \in \mathcal{T}(w)$. Moreover, it should be noted that if \bar{t} and \bar{t}' satisfy $\theta_{i, i'}(k) = 0$ then they are elements of $\mathcal{K}(n)_i$ and satisfy either $\bar{t}' \prec_i \bar{t}$ or $\bar{t} \prec_i \bar{t}'$. Furthermore, if they satisfy $\mu(\bar{t}') + \theta_{i, i'}(k) = \mu(\bar{t})$ then \bar{t}_1 and \bar{t}'_1 are elements of $\mathcal{K}(n)_{i_1}$ and satisfy either $\bar{t}'_1 \prec_{i_1} \bar{t}_1$ or $\bar{t}_1 \prec_{i_1} \bar{t}'_1$. Now, we define the finite set Γ_τ .

Definition 2.4 The finite subset Γ_τ of the root system Φ_N is defined by

$$\Gamma_\tau := \Gamma_\tau^{(1)} \cup \Gamma_\tau^{(2)} \subset \Phi_N.$$

Example 2.5 Now we consider the orbit data $\tau = (n, \sigma, \kappa)$, where $n = 2$, $\sigma = \text{id}$, $\kappa(1, \iota) = 3$ and $\kappa(2, \iota) = 4$ for any $\iota = 1, 2, 3$. Then τ satisfies the assumptions in Theorem 1.5, and thus w_τ is realized by the pair (π_τ, F_τ) , where $\pi_\tau : X_\tau \rightarrow \mathbb{P}^2$ is a blowup of 21 points. A little calculation shows that the entropy of F_τ is given by $h_{\text{top}}(F_\tau) = \log \lambda(F_\tau^*) \approx 1.35442759$, where $\lambda(F_\tau^*) \approx 3.87454251$ is a root of the equation $t^6 - 4t^5 + t^4 - 2t^3 + t^2 - 4t + 1 = 0$. Moreover, the element $w_\tau \in W_{21}$ admits periodic roots $\alpha_{\iota, \tau}^0$ with $i = i' \in \{1, 2\}$ and $\iota \neq \iota' \in \{1, 2, 3\}$, which are not contained in Γ_τ . Therefore, the automorphism F_τ does not appear in the paper of McMullen [10]. On the other hand, for any data $\hat{\tau} = (1, \hat{\sigma}, \hat{\kappa})$, let $F_{\hat{\tau}} : X_{\hat{\tau}} \rightarrow X_{\hat{\tau}}$ be an automorphism that Diller in [5] constructs from a single quadratic map preserving a cuspidal cubic. We claim that, for any $m \geq 1$, F_τ is not topologically conjugate to $F_{\hat{\tau}}^m$. Indeed, assume the contrary that F_τ is topologically conjugate to $F_{\hat{\tau}}^m$ for some data $\hat{\tau}$ and $m \geq 1$. Since X_τ is obtained by blowing up 21 points, so is $X_{\hat{\tau}}$, which means that $\sum_{\iota=1}^3 \hat{\kappa}(1, \iota) = 21$. Thus, there are 570 possibilities for $\hat{\tau}$. Moreover, one has $\lambda(F_\tau^*) = \lambda(F_{\hat{\tau}}^*)^m$. However, with the help of a computer, it may be easily seen that there are no data $\hat{\tau}$ and $m \geq 1$ satisfying the conditions $\sum_{\iota=1}^3 \hat{\kappa}(1, \iota) = 21$ and $\lambda(F_\tau^*) = \lambda(F_{\hat{\tau}}^*)^m$. Therefore our claim is proved.

We conclude this section by establishing the following proposition.

Proposition 2.6 *For any $w \in W_N$, there is an orbit data $\tau = (n, \sigma, \kappa)$ with $\sum_{\bar{\tau} \in \mathcal{K}(n)} \kappa(\bar{\tau}) = N$ such that $w = w_\tau : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$ under some identification $\{e_j \mid j = 1, \dots, N\} = \{e_{\bar{\tau}}^k \mid \bar{\tau} \in \mathcal{K}(n), k = 1, \dots, \kappa(\bar{\tau})\}$.*

Proof. Since w is an element of W_N , it can be expressed as

$$w = \wp_0 \cdot \rho_0 \cdot \wp_1 \cdots \rho_0 \cdot \wp_{m-1} \cdot \rho_0 \cdot \wp_m,$$

where \wp_k is a permutation of $(e_j)_{j=1}^N$. The expression can be written as

$$w = (\wp_0 \cdots \wp_m) \cdot \{(\wp_1 \cdots \wp_m)^{-1} \cdot \rho_0 \cdot (\wp_1 \cdots \wp_m)\} \cdots \{(\wp_{m-1} \cdot \wp_m)^{-1} \cdot \rho_0 \cdot (\wp_{m-1} \cdot \wp_m)\} \cdot \{\wp_m^{-1} \cdot \rho_0 \cdot \wp_m\}.$$

Let $\wp := \wp_0 \cdots \wp_m$ be the permutation on the basis elements $(e_j)_{j=1}^N$, and let $\hat{m} \geq 0$ be the number of orbits $\{\wp^k(e_j) \mid k \geq 0\}$ not containing $(\wp_i \cdots \wp_m)^{-1}(e_\iota)$ for any $i = 1, \dots, m$ and $\iota = 1, 2, 3$. Then put $q_{k+2\hat{m}} := (\wp_k \cdots \wp_m)^{-1} \cdot \rho_0 \cdot (\wp_k \cdots \wp_m)$ and $e_{2\hat{m}+i, \iota}^1 := (\wp_i \cdots \wp_m)^{-1}(e_\iota)$ for $\iota = 1, 2, 3$. Moreover, there are functions $\hat{\kappa} : \{1, \dots, \hat{m}\} \rightarrow \mathbb{Z}_{\geq 1}$ and $\check{\kappa} : \mathcal{K}(m; \hat{m}) := \{(2\hat{m} + i, \iota) \mid i = 1, \dots, m, \iota = 1, 2, 3\} \rightarrow \mathbb{Z}_{\geq 0}$, and a permutation $\hat{\sigma}$ of $\mathcal{K}(m; \hat{m})$, such that the following relations hold:

- if $e_{2\hat{m}+i, \iota}^1 = e_{2\hat{m}+i'', \iota''}^1$ for some $i < i''$, then $\check{\kappa}(2\hat{m} + i, \iota) = 0$ and $\hat{\sigma}(2\hat{m} + i, \iota) = (2\hat{m} + i', \iota')$, where $(2\hat{m} + i', \iota')$ is determined by the relations $e_{2\hat{m}+i, \iota}^1 = e_{2\hat{m}+i', \iota'}^1$ and $i' = \min\{i'' > i \mid e_{2\hat{m}+i, \iota}^1 = e_{2\hat{m}+i'', \iota''}^1\}$,
- after reordering $(e_j)_{j=1}^N$, the permutation \wp is expressed as

$$\wp : \begin{cases} e_{2j,3}^k & \mapsto e_{2j,3}^{k-1} & (j = 1, \dots, \hat{m}, \quad k \in \mathbb{Z}/\hat{\kappa}(j)\mathbb{Z}) \\ e_{\hat{\sigma}(2\hat{m}+i, \iota)}^1 & \mapsto e_{2\hat{m}+i, \iota}^{\check{\kappa}(2\hat{m}+i, \iota)} & (i = 1, \dots, m, \quad \check{\kappa}(2\hat{m} + i, \iota) \geq 1) \\ e_{2\hat{m}+i, \iota}^k & \mapsto e_{2\hat{m}+i, \iota}^{k-1} & (2 \leq k \leq \check{\kappa}(2\hat{m} + i, \iota), i = 1, \dots, m). \end{cases}$$

Then the data $\tau = (n, \sigma, \kappa)$ is defined by $n := m + 2\hat{m}$ and

$$\sigma(i, \iota) := \begin{cases} (i+1, \iota) & (\text{either } \iota = 1, 2 \text{ and } i = 1, \dots, 2\hat{m}, \text{ or } \iota = 3 \text{ and } i = 1, 3, \dots, 2\hat{m}-1) \\ (i-1, \iota) & (\iota = 3, i = 2, 4, \dots, 2\hat{m}) \\ (1, \iota') & (\hat{\sigma}(i, \iota) = (2\hat{m}+1, \iota'), \iota' = 1, 2) \\ \hat{\sigma}(i, \iota) & (\text{otherwise}), \end{cases}$$

$$\kappa(i, \iota) := \begin{cases} 0 & (\text{either } \iota = 1, 2 \text{ and } i = 1, \dots, 2\hat{m}, \text{ or } \iota = 3 \text{ and } i = 1, 3, \dots, 2\hat{m}-1) \\ \hat{\kappa}(i/2) & (\iota = 3, i = 2, 4, \dots, 2\hat{m}) \\ \check{\kappa}(i, \iota) & (\text{otherwise}), \end{cases}$$

which gives expressions $\wp = r_\tau \circ q_1 \circ \dots \circ q_{2\hat{m}}$ and $w = w_\tau = r_\tau \circ q_1 \circ \dots \circ q_n$. Thus the proposition is established. \square

3 The Weyl Group Action

Let us consider the eigenvalues of an automorphism $w : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$ in W_N . If the spectral radius $\lambda(w)$ of w is strictly greater than 1, in other words w admits an eigenvalue d that is not a root of unity, then the eigenvector \bar{v}_d of w corresponding to d determines whether $z \in \mathbb{Z}^{1,N}$ is a periodic vector of w , as is stated in Lemma 3.1. Moreover, we find the coefficients of \bar{v}_d by expressing w as $w = w_\tau$ for an orbit data τ .

Let $w : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$ be a lattice automorphism in W_N . It is known that the characteristic polynomial $\chi_w(t)$ of w can be expressed as

$$\chi_w(t) = \begin{cases} R_w(t) & (\lambda(w) = 1) \\ R_w(t)S_w(t) & (\lambda(w) > 1), \end{cases}$$

where $R_w(t)$ is a product of cyclotomic polynomials, and $S_w(t)$ is a Salem polynomial, namely, the minimal polynomial of a Salem number. Here, a *Salem number* is an algebraic integer $\delta > 1$ whose conjugates other than δ satisfy $|\delta'| \leq 1$ and include $1/\delta < 1$. Therefore, if $w \in W_N$ satisfies $\lambda(w) > 1$, then its unique eigenvalue δ with $|\delta| > 1$ is a Salem number $\delta = \lambda(w) > 1$.

Now assume that $\lambda(w) > 1$. Then there is a direct sum decomposition of the real vector space $\mathbb{R}^{1,N} := \mathbb{Z}^{1,N} \otimes_{\mathbb{Z}} \mathbb{R}$ as

$$\mathbb{R}^{1,N} = V_w \oplus V_w^c,$$

such that the decomposition is preserved by w , and $S_w(t)$ and $R_w(t)$ are the characteristic polynomials of $w|_{V_w}$ and $w|_{V_w^c}$, respectively. We notice that V_w^c is the orthogonal complement of V_w with respect to the Lorentz inner product. Moreover, let ℓ_w be the minimal positive integer satisfying $d^{\ell_w} = 1$ for any root d of the equation $R_w(t) = 0$. Then we have the following lemma.

Lemma 3.1 *Assume that $\delta = \lambda(w) > 1$, and let $\check{\delta}$ be an eigenvalue of w that is not a root of unity. Then, for a vector $z \in \mathbb{Z}^{1,N}$, the following are equivalent.*

- (1) $(z, \bar{v}_{\check{\delta}}) = 0$, where \bar{v}_d is the eigenvector of w corresponding to an eigenvalue d .
- (2) $z \in V_w^c \cap \mathbb{Z}^{1,N}$.
- (3) z is a periodic vector of w with period ℓ_w .

Proof. (1) \implies (2). First, we notice that \bar{v}_d can be chosen so that $\bar{v}_d \in \mathbb{Z}^{1,N} \otimes_{\mathbb{Z}} \mathbb{Z}[d]$. Thus, the coefficient of e_i in $\bar{v}_{\check{\delta}}$, and thus that in $\bar{v}_{\delta'}$ for any conjugate δ' , are expressed as $(\bar{v}_{\check{\delta}})_i = v_i(\check{\delta})$ and $(\bar{v}_{\delta'})_i = v_i(\delta')$ for some $v_i(x) \in \mathbb{Z}[x]$. Since $z \in \mathbb{Z}^{1,N}$, we have $\sum z_i \cdot v_i(x) \in \mathbb{Z}[x]$, and so $(z, \bar{v}_{\delta'}) = \sum z_i \cdot v_i(\delta') = 0$ from the relation $(z, \bar{v}_{\check{\delta}}) = \sum z_i \cdot v_i(\check{\delta}) = 0$. Thus it follows that $z \in V_w^c \cap \mathbb{Z}^{1,N}$.

(2) \implies (3). For any eigenvalues d, d' , we have

$$(\bar{v}_d, \bar{v}_{d'}) = (w(\bar{v}_d), w(\bar{v}_{d'})) = d \cdot d' \cdot (\bar{v}_d, \bar{v}_{d'}),$$

which means that $(\bar{v}_d, \bar{v}_{d'}) = 0$ if $d \cdot d' \neq 1$. In particular, one has $(\bar{v}_{\delta}, \bar{v}_{\delta}) = (\bar{v}_{1/\delta}, \bar{v}_{1/\delta}) = 0$. Moreover, since $\bar{v}_{\delta}, \bar{v}_{1/\delta} \in \mathbb{R}^{1,N}$ are linearly independent over \mathbb{R} , $(\bar{v}_{\delta}, \bar{v}_{1/\delta})$ is nonzero, and thus either $(\bar{v}_{\delta} + \bar{v}_{1/\delta}, \bar{v}_{\delta} + \bar{v}_{1/\delta})$ or $(\bar{v}_{\delta} - \bar{v}_{1/\delta}, \bar{v}_{\delta} - \bar{v}_{1/\delta})$ is positive. As $\mathbb{R}^{1,N}$ has signature $(1, N)$ and V_w has signature $(1, s)$ for some $s \geq 1$, V_w^c is negative definite. This shows that $w|_{V_w^c}$ has finite order. Since any eigenvalue d of $w|_{V_w^c}$ satisfies $d^{\ell_w} = 1$, we have $w^{\ell_w}(z) = z$.

(3) \implies (1). Assume that $w^{\ell_w}(z) = z$. We express z as $z = z' + z''$ for some $z' \in V_w$ and $z'' \in V_w^c$, and then express z' as $z' = \sum_{S_w(d)=0} z_d \cdot \bar{v}_d$ for some $z_d \in \mathbb{C}$. Under the assumption that $w^{\ell_w}(z) = z$, one has $\sum_{S_w(d)=0} z_d \cdot \bar{v}_d = z' = w^{\ell_w}(z') = \sum_{S_w(d)=0} d^{\ell_w} \cdot z_d \cdot \bar{v}_d$. This means that $z_d = d^{\ell_w} \cdot z_d$ for any d with $S_w(d) = 0$. Since d is not a root of unity, z_d is zero for any d . Therefore, we have $z' = 0$ and $z = z'' \in V_w^c$, and the assertion is established. \square

Remark 3.2 For an orbit data τ , the positive integer ℓ_{τ} (see Definition 2.2) satisfies

$$\ell_{\tau} = \ell_{w_{\tau}}.$$

Next, we describe eigenvectors of the lattice automorphism $w_{\tau} : \mathbb{Z}^{1,N} \rightarrow \mathbb{Z}^{1,N}$ for a given orbit data $\tau = (n, \sigma, \kappa)$. To this end, we consider the system of equations

$$v_{i,1} + v_{i,2} + v_{i,3} = - \sum_{k=1}^{i-1} s_k + (d-2) \cdot s_i - d \sum_{k=i+1}^n s_k, \quad (1 \leq i \leq n), \quad (13)$$

$$v_{\bar{\iota}_1} = d^{\kappa(\bar{\iota})} \cdot v_{\bar{\iota}} + (d-1) \cdot s_{i_1} \quad (\bar{\iota} \in \mathcal{K}(n)), \quad (14)$$

where $d \in \mathbb{C} \setminus \{0\}$, $v = (v_{\bar{\iota}})_{\bar{\iota} \in \mathcal{K}(n)} \in \mathbb{C}^{3n}$, $s = (s_r)_{r=1}^n \in \mathbb{C}^n$, and $\bar{\iota}_m = (i_m, \iota_m) = \sigma^m(\bar{\iota})$ for $\bar{\iota} = (i, \iota) \in \mathcal{K}(n)$.

Proposition 3.3 Let τ be an orbit data, and \bar{v} be a vector in $L_{\tau} \otimes_{\mathbb{Z}} \mathbb{C}$ expressed as

$$\bar{v} = v_0 \cdot e_0 + \sum v_{\bar{\iota}}^k \cdot e_{\bar{\iota}}^k \in L_{\tau} \otimes_{\mathbb{Z}} \mathbb{C}.$$

If \bar{v} is an eigenvector $\bar{v} = \bar{v}_d$ of w_{τ} corresponding to an eigenvalue d different from 1, then there is a unique pair $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times \mathbb{C}^n$ such that the following conditions hold:

- (1) (d, v, s) satisfies equations (13) and (14).
- (2) $v_{\bar{\tau}}^k = d^{k-1} \cdot v_{\bar{\tau}}$ for any $\bar{\tau} \in \mathcal{K}(n)$ and $1 \leq k \leq \kappa(\bar{\tau})$.
- (3) $v_0 = k(s)$, where $k(s)$ is given by

$$k(s) := \sum_{k=1}^n s_k. \quad (15)$$

Conversely, if \bar{v} satisfies conditions (1)–(3) for some triplet $(d, v, s) \in (\mathbb{C} \setminus \{0, 1\}) \times (\mathbb{C}^{3n} \setminus \{0\}) \times \mathbb{C}^n$, then \bar{v} is an eigenvector $\bar{v} = \bar{v}_d$ of w_τ corresponding to an eigenvalue d .

Proof. Assume that \bar{v} is an eigenvector corresponding to $d \neq 1$. For any $1 \leq k \leq \kappa(\bar{\tau}) - 1$, the coefficient of $e_{\bar{\tau}}^k$ in $w_\tau(\bar{v})$ is $v_{\bar{\tau}}^{k+1}$. Hence, one has $v_{\bar{\tau}}^{k+1} = d \cdot v_{\bar{\tau}}^k$, and $v_{\bar{\tau}}^k = d^{k-1} \cdot v_{\bar{\tau}}^1$. Moreover, we put $v_{\bar{\tau}} := v_{\bar{\tau}}^1$ and choose s_1, \dots, s_n so that $v_0 = \sum_{i=1}^n s_i$ and $v_j = -\sum_{i=1}^{j-1} s_i + (d-2) \cdot s_j - d \sum_{i=j+1}^n s_i$ for $2 \leq j \leq n$, where $v_j := v_{j,1} + v_{j,2} + v_{j,3}$, and $v_{\bar{\tau}} := v_{\bar{\tau},1} - (d-1) \cdot s_{i_1}$ if $\kappa(\bar{\tau}) = 0$. Note that s_1, \dots, s_n are determined uniquely since $d \neq 1$. Now we claim that $v_1 = (d-2) \cdot s_1 - d \sum_{i=2}^n s_i$, and that the following relation holds for $1 \leq j \leq n+1$:

$$q_j \circ \dots \circ q_n(\bar{v}) = v^j \cdot e_0 + \sum_{\substack{i < j \\ \kappa(\bar{\tau}) \geq 1}} v_{\bar{\tau}} \cdot e_{\bar{\tau}}^1 + \sum_{\substack{i < j \leq i_1 \\ \kappa(\bar{\tau}) = 0}} v_{\bar{\tau}} \cdot e_{\bar{\tau}}^1 + \sum_{\substack{j \leq i \\ \kappa(\bar{\tau}-1) \geq 1}} \left\{ v_{\bar{\tau}} - (d-1) \cdot s_i \right\} \cdot e_{\bar{\tau}}^1 + \sum_{k \geq 2} d^{k-1} \cdot v_{\bar{\tau}} \cdot e_{\bar{\tau}}^k, \quad (16)$$

where $v^j := \sum_{i=1}^{j-1} s_i + d \sum_{i=j}^n s_i$. Indeed, if $j = n+1$, the relation is trivial. Assume that the relation holds when $j+1 \geq 2$. Then the automorphism q_j changes only the coefficients v^{j+1} and $v_{j,\iota}$ in $q_{j+1} \circ \dots \circ q_n(\bar{v})$ as follows:

$$q_j \left(v^{j+1} \cdot e_0 + \sum_{\iota=1}^3 v_{j,\iota} \cdot e_{(j,\iota)\tau}^1 \right) = (2v^{j+1} + v_j) \cdot e_0 - \sum_{\iota=1}^3 (v^{j+1} + v_{j,\iota'} + v_{j,\iota''}) \cdot e_{(j,\iota)\tau}^1,$$

where $\{\iota, \iota', \iota''\} = \{1, 2, 3\}$. Therefore, when $j \geq 2$, equation (16) holds from the facts that $2v^{j+1} + v_j = v^j$, $v^{j+1} + v_{j,\iota'} + v_{j,\iota''} = (d-1) \cdot s_j - v_{j,\iota}$, and $\{v_{j,\iota} - (d-1) \cdot s_j\} \cdot e_{(j,\iota)\tau}^1 = v_{j-1,\iota-1} \cdot e_{(j-1,\iota-1)\tau}^1$ if $\kappa(j-1, \iota-1) = 0$. Moreover, since the coefficient of e_0 in $w_\tau(\bar{v})$ is $d \cdot v_0$ and r_τ fixes e_0 , the coefficient of e_0 in $q_1 \circ \dots \circ q_n(\bar{v})$ is expressed as $2v^2 + v_1 = d \cdot v_0 = v^1$, which yields $v_1 = (d-2) \cdot s_1 - d \sum_{i=2}^n s_i$. Thus, the coefficient of $e_{1,\iota}$ in $q_1 \circ \dots \circ q_n(\bar{v})$ is given by $v_{1,\iota} - (d-1) \cdot s_1$ and equation (16) holds when $j = 1$. The claim follows from these observations. In particular, (d, v, s) satisfies equation (13).

By the above claim, we have

$$q_1 \circ \dots \circ q_n(\bar{v}) = v^1 \cdot e_0 + \sum_{\kappa(\bar{\tau}) \geq 1} \left\{ v_{\bar{\tau},1} - (d-1) \cdot s_{i_1} \right\} \cdot e_{\sigma_\tau(\bar{\tau})}^1 + \sum_{k \geq 2} d^{k-1} \cdot v_{\bar{\tau}} \cdot e_{\bar{\tau}}^k,$$

as $(\bar{\tau}_1)_\tau = \sigma_\tau(\bar{\tau})$. Thus, the coefficient of e_0 in $w_\tau(\bar{v})$ is $v^1 = d \cdot k(s)$. Similarly, the coefficient of $e_{\bar{\tau}}^{\kappa(\bar{\tau})}$ in $w_\tau(\bar{v})$ is given by $v_{\bar{\tau},1} - (d-1) \cdot s_{i_1}$. This means that $v_{\bar{\tau},1} - (d-1) \cdot s_{i_1} = d \cdot v_{\bar{\tau}}^{\kappa(\bar{\tau})} = d^{\kappa(\bar{\tau})} \cdot v_{\bar{\tau}}$ and that (d, v, s) satisfies equation (14). Moreover we have $v \neq 0$, since if $v = 0$, then one has $s = 0$ from (14), and so $\bar{v} = \bar{v}_d = 0$, which is a contradiction.

The rest of the statement in the proposition immediately follows from the above discussion, and the proof is complete. \square

Now assume that d is not a root of unity. Then equation (14) is equivalent to the expression

$$v_{\bar{\tau}} = v_{\bar{\tau}}(d) = -\frac{d^{\varepsilon_{|\bar{\tau}|}} \cdot (d-1)}{d^{\varepsilon_{|\bar{\tau}|}} - 1} \left(d^{-\varepsilon_1} \cdot s_{i_1} + d^{-\varepsilon_2} \cdot s_{i_2} + \cdots + d^{-\varepsilon_{|\bar{\tau}|}} \cdot s_{i_{|\bar{\tau}|}} \right), \quad (17)$$

where $|\bar{\tau}| := \#\{\bar{\tau}_m \mid m \geq 0\}$ and $\varepsilon_r := \varepsilon_r(\bar{\tau}) = \sum_{k=0}^{r-1} \kappa(\bar{\tau}_k)$. Let $\bar{c}_{\bar{\tau},j}(d)$ and $c_{i,j}(d)$ be the polynomials of d defined by

$$v_{\bar{\tau}}(d) = \sum_{j=1}^n \bar{c}_{\bar{\tau},j}(d) \cdot s_j, \quad (18)$$

$$v_i(d) := v_{i,1}(d) + v_{i,2}(d) + v_{i,3}(d) = -\sum_{j=1}^n c_{i,j}(d) \cdot s_j, \quad (19)$$

and let $\mathcal{A}_n(d, x)$ be the $n \times n$ matrix having the (i, j) -th entry:

$$\mathcal{A}_n(d, x)_{i,j} = \begin{cases} d - 2 + x_{i,i} & (i = j) \\ -1 + x_{i,j} & (i > j) \\ -d + x_{i,j} & (i < j) \end{cases} \quad (20)$$

with $x = (x_1, \dots, x_n) = (x_{ij}) \in M_n(\mathbb{R})$. Then equations (13) and (17) yield

$$\mathcal{A}_\tau(d) s = 0, \quad s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad (21)$$

where $\mathcal{A}_\tau(d) := \mathcal{A}_n(d, c(d))$ with $c(d) := (c_{i,j}(d))$. Finally, let $\chi_\tau(d)$ be the determinant $|\mathcal{A}_\tau(d)|$ of the matrix $\mathcal{A}_\tau(d)$.

Corollary 3.4 *Assume that d is not a root of unity. Then,*

- (1) *d is a root of $\chi_\tau(t) = 0$ if and only if d is a root of $S_{w_\tau}(t) = 0$.*
- (2) *If d is a root of $S_{w_\tau}(t) = 0$, then there is a unique solution $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ of (13) and (14), up to a constant multiple. Conversely, if there is a solution $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ of (13) and (14), then d is a root of $S_{w_\tau}(t) = 0$.*
- (3) *If $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ satisfies (13) and (14), then v and s are nonzero, and s is a unique solution of (21). Conversely, if $s \neq 0$ is a solution of (21), then (v, s) satisfies (13) and (14), where $v \neq 0$ is given in (17).*

Proof. First, we notice that if $(v, s) \neq (0, 0) \in \mathbb{C}^{3n} \times \mathbb{C}^n$ satisfies (13) and (14), then we have $v \neq 0$ and $s \neq 0$. Indeed, if $v = 0$ then $s = 0$ from (14), and if $s = 0$ then $v = 0$ from (17). Now assume that d is a root of $\chi_\tau(t) = 0$ that is not a root of unity. Then there is a solution $s \neq 0$ of (21). Moreover, (v, s) satisfies (13) and (14), where v is given in (17), and thus is nonzero. By Proposition 3.3, there is an eigenvector \bar{v}_d of w_τ , which shows that $S_{w_\tau}(d) = 0$.

Conversely, assume that d is a root of $S_{w_\tau}(t) = 0$. Since the eigenvector \bar{v}_d is unique, there is a unique solution $(v, s) \in (\mathbb{C}^{3n} \setminus \{0\}) \times \mathbb{C}^n$ of (13) and (14), which yields $s \neq 0$. Moreover, s is a unique solution of (21), up to a constant multiple. Indeed, if $s \neq s'$ are solutions of (21), then there are solutions $(v, s) \neq (v', s')$ of (13) and (14), which is a contradiction. Thus d is a root of $\chi_\tau(t) = 0$. \square

4 Construction of Rational Surface Automorphisms

In this section, we develop a method for constructing a rational surface automorphism from a composition $f = f_n \circ \cdots \circ f_1 : Y \rightarrow Y$ of general birational maps $f_i : Y_{i-1} \rightarrow Y_i$ between smooth rational surfaces and a generalized orbit data τ . If the data τ is compatible with the maps $\bar{f} = (f_1, \dots, f_n)$, f lifts to an automorphism $F : X \rightarrow X$ through a blowup $\pi : X \rightarrow Y$. Moreover, in the special case where \bar{f} are quadratic birational maps on \mathbb{P}^2 and τ is an original orbit data, we calculate the action $F^* : H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ of the automorphism F , which shows that w_τ is realized by (π, F) .

First we collect some terminology about complex surfaces (see also [1]). Let Y be a smooth projective irreducible surface, and $\pi_y : Y_y \rightarrow Y$ be the blowup of a point y on Y with the exceptional divisor E_y in Y_y . Then each point on E_y is called a *point in the first infinitesimal neighbourhood of y on Y* . Moreover, for $i > 0$, we inductively define a *point in the i -th infinitesimal neighbourhood of y on Y* as a point in the first infinitesimal neighbourhood of some point in the $(i-1)$ -th infinitesimal neighbourhood of y on Y , where a point in the 0-th infinitesimal neighbourhood of y is interpreted as y itself. A point in the i -th infinitesimal neighbourhood of y on Y for some $i > 0$ is called a *point infinitely near to y on Y* , or an *infinitely near point on Y* . We sometimes call the points on Y *the proper points* to distinguish them from the infinitely near points. In what follows, a point $y' \in Y$ means that either it is proper on Y or it is infinitely near to some proper point on Y , and $y_1 = y_2$ means that they are both in the same infinitesimal neighbourhood of a proper point and are equal. Moreover, through the blowup $\pi_y : Y_y \rightarrow Y$, any point $y' \in Y_y$ is identified with $\pi_y(y')$, and $\pi_y(y')$ is also denoted by $y' \in Y$. Then, a point in the i -th infinitesimal neighbourhood of y' on Y_y is in the $(i-1)$ -th infinitesimal neighbourhood of y on Y or in the i -th infinitesimal neighbourhood of y' on Y , according to whether $y' \in E_y$ or $y' \notin E_y$. For two points y_1, y_2 of Y , we write $y_1 < y_2$ if y_2 is infinitely near to y_1 , and write $y_1 \approx y_2$ if either $y_1 = y_2$ or $y_1 < y_2$ or $y_1 > y_2$. A *cluster* $I \subset X$ is a finite set of proper or infinitely near points on X such that if $y \in I$ and $y' < y$, then $y' \in I$. From the cluster $I = \{y_1, \dots, y_N\}$, one can construct the blowup $\pi_I : \tilde{Y} \rightarrow Y$ of the points in I , that is, the composition

$$\pi_I : \tilde{Y} = Y_N \rightarrow Y_{N-1} \rightarrow \cdots \rightarrow Y_0 = Y \quad (22)$$

of blowups $\pi_i : Y_i \rightarrow Y_{i-1}$ of a point $y_{k_i} \in I$ such that if $y_{k_i} < y_{k_j}$ then $i < j$. Note that the surface \tilde{Y} is determined uniquely by the cluster I , namely, if $\pi'_I : \tilde{Y}' \rightarrow Y$ is constructed from another ordering $(y_{k'_1}, \dots, y_{k'_N})$ of I , then there is a unique isomorphism $g : \tilde{Y} \rightarrow \tilde{Y}'$ such that $\pi_I = \pi'_I \circ g : \tilde{Y} \rightarrow Y$.

An example of clusters is indeterminacy sets of birational surface maps. Let Y_+ and Y_- be smooth surfaces, and $f : Y_+ \rightarrow Y_-$ be a birational map with its inverse $f^{-1} : Y_- \rightarrow Y_+$. In general, $f^{\pm 1}$ may admit clusters $I(f^{\pm 1})$ in Y_{\pm} on which $f^{\pm 1}$ are not defined. The clusters $I(f^{\pm 1})$ are called the *indeterminacy sets* of $f^{\pm 1}$. Moreover, any blowup $\pi_+ : \tilde{Y}_+ \rightarrow Y_+$ of a cluster $I \subset Y_+$ uniquely lifts $f : Y_+ \rightarrow Y_-$ to $\tilde{f} = f \circ \pi_+ : \tilde{Y}_+ \rightarrow Y_-$, which determines the point $\tilde{f}(y)$ for any proper point $y \in \tilde{Y}_+ \setminus I(\tilde{f})$. When regarding y as a point on Y_+ , we write $\tilde{f}(y) = \tilde{f}(y)$. In this setting, the following properties hold:

- $I(\tilde{f}) = I(f) \setminus I$.
- If $y < y' \in Y_+$ and $y \notin I(f)$, then $y' \notin I(f)$ and $f(y) < f(y')$.
- If $y \notin I(f)$ and $f(y) \approx y' \in I(f^{-1})$, then $y' < f(y)$.
- If a proper point $y \notin I(f)$ satisfies $f(y) \not\approx y'$ for any $y' \in I(f^{-1})$, then $f(y)$ is also a proper point on Y_- .

Now we consider a smooth rational surface X , that is, a surface birationally equivalent to \mathbb{P}^2 , and an automorphism $F : X \rightarrow X$ of X . By theorems of Gromov and Yomdin, the topological entropy of F is given by $h_{\text{top}}(F) = \log \lambda(F^*)$, where $\lambda(F^*)$ is the spectral radius of the action $F^* : H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ on the cohomology group. In this paper, we are interested in the case where $F : X \rightarrow X$ has positive entropy $h_{\text{top}}(F) > 0$ or, in other words, $\lambda(F^*) > 1$. Then, the surface X is characterized as follows (see [9, 11]).

Proposition 4.1 *If X admits an automorphism $F : X \rightarrow X$ with $\lambda(F^*) > 1$, then there is a birational morphism $\pi : X \rightarrow \mathbb{P}^2$.*

It is known that any birational morphism $\pi : X \rightarrow \mathbb{P}^2$ is expressed as $\pi = \pi_I$ for some cluster $I = \{x_1, \dots, x_N\} \subset \mathbb{P}^2$, where π_I is the blowup of I given in (22) with $Y = \mathbb{P}^2$ and $\tilde{Y} = X$. Then $\pi : X \rightarrow \mathbb{P}^2$ determines an expression of the cohomology group: $H^2(X; \mathbb{Z}) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \dots \oplus \mathbb{Z}[E_N]$, where H is the total transform of a line in \mathbb{P}^2 , and E_i is the total transform of the point x_i . The intersection form on $H^2(X; \mathbb{Z})$ is given by

$$\begin{cases} ([H], [H]) = 1, \\ ([E_i], [E_j]) = -\delta_{i,j}, & (i, j = 1, \dots, N), \\ ([H], [E_i]) = 0, & (i = 1, \dots, N). \end{cases}$$

Therefore $H^2(X; \mathbb{Z})$ is isometric to the Lorentz lattice $\mathbb{Z}^{1,N}$ given in (1). Namely, there is a natural marking isomorphism $\phi_\pi : \mathbb{Z}^{1,N} \rightarrow H^2(X, \mathbb{Z})$, sending the basis as

$$\phi_\pi(e_0) = [H], \quad \phi_\pi(e_i) = [E_i] \quad (i = 1, \dots, N).$$

The marking ϕ_π is isometric and determined uniquely by $\pi : X \rightarrow \mathbb{P}^2$ in the sense that if ϕ_π and ϕ'_π are markings determined by π , then there is an element $\wp \in \langle \rho_1, \dots, \rho_{N-1} \rangle$, acting by a permutation on the basis elements (e_1, \dots, e_N) , such that $\phi_\pi = \phi'_\pi \circ \wp$. The following proposition indicates a role of the Weyl group W_N (see [6, 8, 12]).

Proposition 4.2 *For any birational morphism $\pi : X \rightarrow \mathbb{P}^2$ and any automorphism $F : X \rightarrow X$, there is a unique element $w \in W_N$ such that diagram (3) commutes.*

Thus, a pair (π, F) determines w uniquely, up to conjugacy by an element of $\langle \rho_1, \dots, \rho_{N-1} \rangle$. In this case, the element w is said to be *realized* by (π, F) , and the entropy of F is expressed as $h_{\text{top}}(F) = \log \lambda(w)$. Summing up these discussions, we have the following proposition.

Proposition 4.3 *The entropy of any automorphism $F : X \rightarrow X$ on a rational surface X is given by $h_{\text{top}}(F) = \log \lambda$ for some $\lambda \in \Lambda$, where Λ is given in (6).*

Indeed, when $F : X \rightarrow X$ satisfies $\lambda(F^*) = 1$, the entropy of F is expressed as $h_{\text{top}}(F) = \log \lambda(e)$ with the unit element $e \in W_N$.

Remark 4.4 If $\pi : X \rightarrow \mathbb{P}^2$ is a blowup of N points with $N \leq 9$, and $F : X \rightarrow X$ is an automorphism, then it follows that $h_{\text{top}}(F) = 0$ (see e.g. [10]).

Next we turn our attention to a method for constructing rational surface automorphisms. Let Y_1, \dots, Y_n be general smooth rational surfaces, and $\bar{f} := (f_1, \dots, f_n)$ be an n -tuple of birational maps $f_r : Y_{r-1} \rightarrow Y_r$ with $Y_0 := Y_n$, and let $I(f_r) = \{p_{r,1}^+, \dots, p_{r,j_+(r)}^+\} \subset Y_{r-1}$ and $I(f_r^{-1}) = \{p_{r,1}^-, \dots, p_{r,j_-(r)}^-\} \subset Y_r$ be the indeterminacy sets of f_r and of f_r^{-1} , respectively. Put $\mathcal{K}_{\pm}(\bar{f}) := \{\bar{\iota} = (i, \iota) \mid i = 1, \dots, n, \iota = 1, \dots, j_{\pm}(i)\}$ and $\mathcal{K}(\bar{f}) := \mathcal{K}_-(\bar{f})$. Then it turns out that the cardinalities of the sets $\mathcal{K}_{\pm}(\bar{f})$ are the same, that is, $\sum_{r=1}^n j_+(r) = \sum_{r=1}^n j_-(r)$, since $Y_n = Y_0$. Moreover, for $m \geq 0$ and $\bar{\iota} \in \mathcal{K}(\bar{f})$, we inductively put

$$p_{\bar{\iota}}^0 := p_{\bar{\iota}}^- \in Y_i, \quad p_{\bar{\iota}}^m := f_r(p_{\bar{\iota}}^{m-1}) \in Y_r \quad (r \equiv i + m \pmod{n}). \quad (23)$$

Note that a point $p_{\bar{\iota}}^m$ is well-defined if $p_{\bar{\iota}}^{m'} \notin I(f_{r'+1})$ for any $0 \leq m' < m$, and that $p_{\bar{\iota}}^m$ is a point of Y_r if and only if $m = \theta_{i,r}(k) \geq 0$ for some $k \geq 0$, where $\theta_{i,r}(k)$ is given in (8).

To define the concept of realization, let us introduce a *generalized orbit data* $\tau = (n, \sigma, \kappa)$ for \bar{f} consisting of the integer $n \geq 1$, a bijection $\sigma : \mathcal{K}(\bar{f}) \rightarrow \mathcal{K}_+(\bar{f})$ and a function $\kappa : \mathcal{K}(\bar{f}) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\kappa(\bar{\iota}) \geq 1$ provided $i_1 \leq i$, where $(i_1, \iota_1) = \sigma(\bar{\iota})$.

Definition 4.5 Let \bar{f} be an n -tuple of birational maps and $\tau = (n, \sigma, \kappa)$ be a generalized orbit data for \bar{f} . Then \bar{f} is called a *realization* of τ if the following condition holds for any $\bar{\iota} \in \mathcal{K}(\bar{f})$:

$$p_{\bar{\iota}}^m \neq p_{\bar{\iota}'}^+ \quad (0 \leq m < \mu(\bar{\iota}), \bar{\iota}' \in \mathcal{K}_+(\bar{f})), \quad p_{\bar{\iota}}^{\mu(\bar{\iota})} = p_{\sigma(\bar{\iota})}^+. \quad (24)$$

We should notice that in condition (24), two points $p_{\bar{\iota}}^m$ and $p_{\bar{\iota}'}^+$ may satisfy $p_{\bar{\iota}}^m \approx p_{\bar{\iota}'}^+$ but $p_{\bar{\iota}}^m \neq p_{\bar{\iota}'}^+$. From a realization \bar{f} of τ , we construct an automorphism. To this end, we define

Definition 4.6 A pair $(p_{\bar{\iota}}^-, p_{\bar{\iota}'}^+)$ of indeterminacy points of f_i^{-1} and $f_{i'}$ with $\bar{\iota} = (i, \iota) \in \mathcal{K}(\bar{f})$ and $\bar{\iota}' = (i', \iota') \in \mathcal{K}_+(\bar{f})$ is called a *proper pair of \bar{f} with length μ* if the following three conditions hold:

- (1) $p_{\bar{\iota}}^-$ and $p_{\bar{\iota}'}^+$ are proper points on Y_i and $Y_{i'-1}$, respectively.
- (2) $p_{\bar{\iota}}^m \not\approx p_{r+1,j}^+$ for any $0 \leq m = \theta_{i,r}(k) < \mu$, and $p_{\bar{\iota}}^m \not\approx p_{r,j}^-$ for any $0 < m = \theta_{i,r}(k) \leq \mu$.

$$(3) \ p_{\bar{t}'}^+ = p_{\bar{t}}^\mu.$$

Assume that $(p_{\bar{t}}^-, p_{\bar{t}'}^+)$ is a proper pair. Then one has $p_{\bar{t}}^m \not\approx p_{\bar{t}}^{m'}$ when $m \neq m'$. Indeed, if $p_{\bar{t}}^m \approx p_{\bar{t}}^{m+\hat{m}}$ for some $\hat{m} > 0$, then one has $p_{\bar{t}}^{\mu-\hat{m}} \approx p_{\bar{t}}^\mu = p_{\bar{t}'}^+$, which is a contradiction. Let $\pi_r : Y_r' \rightarrow Y_r$ be the blowup of distinct proper points $(p_{\bar{t}}^m)$ with $0 \leq m = \theta_{i,r}(k) \leq \mu$. Then the blowups $\bar{\pi} := (\pi_r)$ lift $f_r : Y_{r-1} \rightarrow Y_r$ to $f_r' := \pi_r^{-1} \circ f_r \circ \pi_{r-1} : Y_{r-1}' \rightarrow Y_r'$ (see Figure 1). We say that $\bar{f}' := (f_1', \dots, f_n')$ is obtained from \bar{f} by the proper pair $(p_{\bar{t}}^-, p_{\bar{t}'}^+)$ through the blowups $\bar{\pi}$. In this case, one has

$$I(f_r') = \begin{cases} I(f_{i'}) \setminus \{p_{\bar{t}'}^+\} & (r = i') \\ I(f_r) & (r \neq i'), \end{cases} \quad I((f_r')^{-1}) = \begin{cases} I(f_i^{-1}) \setminus \{p_{\bar{t}}^-\} & (r = i) \\ I(f_r^{-1}) & (r \neq i), \end{cases} \quad (25)$$

and also $\mathcal{K}_+(\bar{f}') = \mathcal{K}_+(\bar{f}) \setminus \{\bar{t}'\}$ and $\mathcal{K}(\bar{f}') = \mathcal{K}(\bar{f}) \setminus \{\bar{t}\}$.

Lemma 4.7 *An n -tuple \bar{f} of birational maps is a realization of a generalized orbit data $\tau = (n, \sigma, \kappa)$ if and only if there is a total order $\bar{t}^{(1)} \prec \bar{t}^{(2)} \prec \dots \bar{t}^{(\nu)}$ on $\mathcal{K}(\bar{f})$, with $\nu := \sum_{r=1}^n j_+(r) = \sum_{r=1}^n j_-(r)$, such that $(p_{\bar{t}^{(j)}}^-, p_{\sigma(\bar{t}^{(j)})}^+)$ is a proper pair of \bar{g}_{j-1} with length $\mu(\bar{t}^{(j)})$ for any $j = 1, \dots, \nu$, where $\bar{g}_0 := \bar{f}$ and \bar{g}_j is inductively obtained from \bar{g}_{j-1} by the proper pair $(p_{\bar{t}^{(j)}}^-, p_{\sigma(\bar{t}^{(j)})}^+)$. Moreover, $\mu(\bar{t})$ is defined by*

$$\mu(\bar{t}) := \theta_{i, i_1-1}(\kappa(\bar{t})),$$

where $\bar{t}_1 = (i_1, \iota_1) = \sigma(\bar{t})$.

Proof. Assume that \bar{f} is a realization of τ and there are proper pairs $(p_{\bar{t}^{(\ell)}}^-, p_{\sigma(\bar{t}^{(\ell)})}^+)$ of $\bar{g}_{\ell-1} = (g_{\ell-1,r} : Z_{\ell-1,r-1} \rightarrow Z_{\ell-1,r})_{r=1}^n$ with length $\mu(\bar{t}^{(\ell)})$ for $\ell = 1, \dots, j$, where $\bar{g}_\ell = (g_{\ell,r} : Z_{\ell,r-1} \rightarrow Z_{\ell,r})_{r=1}^n$ is obtained from $\bar{g}_{\ell-1}$ by the proper pair $(p_{\bar{t}^{(\ell)}}^-, p_{\sigma(\bar{t}^{(\ell)})}^+)$ through the blowups $\bar{\pi}_\ell = (\pi_{\ell,r} : Z_{\ell,r} \rightarrow Z_{\ell-1,r})_{r=1}^n$. Take an element $\bar{t}^{(j+1)} = (i^{(j+1)}, \iota^{(j+1)}) \in \mathcal{K}(\bar{g}_j) = \mathcal{K}(\bar{f}) \setminus \{\bar{t}^{(1)}, \dots, \bar{t}^{(j)}\}$ such that $p_{\bar{t}^{(j+1)}}^-$ is a proper point of $Z_{j,i^{(j+1)}}$ and

$$\mu(\bar{t}^{(j+1)}) = \min\{\mu(\bar{t}) \mid \bar{t} \in \mathcal{K}(\bar{g}_j) \text{ and } p_{\bar{t}}^- \text{ is a proper point on } Z_{j,i}\}.$$

It is enough to show that $(p_{\bar{t}^{(j+1)}}^-, p_{\sigma(\bar{t}^{(j+1)})}^+)$ is a proper pair of \bar{g}_j with length $\mu(\bar{t}^{(j+1)})$. First, assume the contrary that $p_{\bar{t}^{(j+1)}}^m \approx p_{\bar{t}'}^-$ for some $0 < m \leq \mu(\bar{t}^{(j+1)})$ and $\bar{t}' \in \mathcal{K}(\bar{g}_j)$. Then we have $p_{\bar{t}^{(j+1)}}^m > p_{\bar{t}'}^-$. The minimality of $\mu(\bar{t}^{(j+1)})$ yields $p_{\bar{t}'}^{m'} \notin I(g_{j,i'})$ for any $0 \leq m' < \mu(\bar{t}^{(j+1)}) - m$, and so $p_{\bar{t}^{(j+1)}}^{\mu(\bar{t}^{(j+1)})-m} > p_{\bar{t}'}^{\mu(\bar{t}^{(j+1)})-m}$. Since $p_{\bar{t}^{(j+1)}}^{\mu(\bar{t}^{(j+1)})} = p_{\sigma(\bar{t}^{(j+1)})}^+ \in I(g_{j,i_1^{(j+1)}})$ and $I(g_{j,i_1^{(j+1)}})$ is a cluster, $p_{\bar{t}'}^{\mu(\bar{t}^{(j+1)})-m}$ is also an element of $I(g_{j,i_1^{(j+1)}})$ and thus is equal to $p_{\sigma(\bar{t}')}^+$. This means that $\mu(\bar{t}') = \mu(\bar{t}^{(j+1)}) - m < \mu(\bar{t}^{(j+1)})$, which contradicts the assumption that $\mu(\bar{t}^{(j+1)})$ is minimal. Thus, we have $p_{\bar{t}^{(j+1)}}^m \not\approx p_{\bar{t}'}^-$ for any $0 < m \leq \mu(\bar{t}^{(j+1)})$ and $\bar{t}' \in \mathcal{K}(\bar{g}_j)$. In particular, $p_{\bar{t}^{(j+1)}}^m$ is proper on $Z_{j,r}$ with $r \equiv i^{(j+1)} + m \pmod{n}$. Since $p_{\bar{t}^{(j+1)}}^m \neq p_{\bar{t}'}^+$ for any $0 \leq m < \mu(\bar{t})$ and $I(g_{j,i'})$ is a cluster, one has $p_{\bar{t}^{(j+1)}}^m \not\approx p_{\bar{t}'}^+$ for any $0 \leq m < \mu(\bar{t})$. Moreover, as $p_{\bar{t}^{(j+1)}}^{\mu(\bar{t}^{(j+1)})} = p_{\sigma(\bar{t}^{(j+1)})}^+$ and $p_{\bar{t}^{(j+1)}}^{\mu(\bar{t}^{(j+1)})}$ is proper, the point $p_{\sigma(\bar{t}^{(j+1)})}^+$ is also proper. Therefore, $(p_{\bar{t}^{(j+1)}}^-, p_{\sigma(\bar{t}^{(j+1)})}^+)$ is a proper pair of \bar{g}_j with length $\mu(\bar{t}^{(j+1)})$.

Conversely, assume that there is a total order $\bar{t}^{(1)} \prec \bar{t}^{(2)} \prec \dots \bar{t}^{(\nu)}$ on $\mathcal{K}(\bar{f})$ such that $(p_{\bar{t}^{(j)}}^-, p_{\sigma(\bar{t}^{(j)})}^+)$ is a proper pair of \bar{g}_{j-1} with length $\mu(\bar{t}^{(j)})$ for each $j = 1, \dots, \nu$. Note that

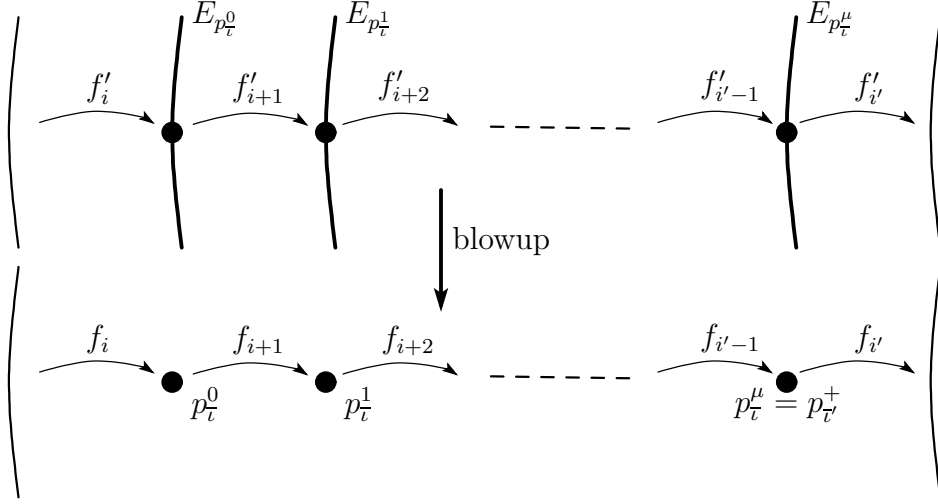


Figure 1: Blowup of indeterminacy points

$I(f_{r+1}) \subset I((\pi_{1,r} \circ \dots \circ \pi_{j-1,r})^{-1}) \cup I(g_{j-1,r+1})$. Therefore we have $p_{\tau(j)}^m \notin I(f_{r+1})$ for any $0 \leq m < \mu(\bar{\tau}^{(j)})$, since $p_{\tau(j)}^m \notin I(g_{j-1,r+1})$ and $p_{\tau(j)}^m > p \in I((\pi_{1,r} \circ \dots \circ \pi_{j-1,r})^{-1})$ if $p_{\tau(j)}^m \approx p$. As $p_{\tau(j)}^{\mu(\bar{\tau}^{(j)})} = p_{\sigma(\tau(j))}^+$, condition (24) holds for any $\bar{\tau}^{(j)} \in \mathcal{K}(\bar{f})$, completing the proof. \square

Assume that \bar{f} is a realization of τ . Then it turns out that the compositions $\pi_r := \pi_{1,r} \circ \dots \circ \pi_{\nu,r} : X_r \rightarrow Y_r$ are blowups of $N = \sum_{\bar{\tau} \in \mathcal{K}(\bar{f})} \kappa(r, \bar{\tau}) = \sum_{\bar{\tau} \in \mathcal{K}(\bar{f})} \kappa(\bar{\tau})$ points $\{p_\tau^m \mid \bar{\tau} \in \mathcal{K}(\bar{f}), 0 \leq m \leq \mu(\bar{\tau}), i + m \equiv r \pmod{n}\}$, where $\kappa(r, \bar{\tau})$ is given by

$$\kappa(r, \bar{\tau}) := \begin{cases} \kappa(\bar{\tau}) + 1 & (\text{if } i < i_1 \text{ and } i \leq r \leq i_1 - 1) \\ \kappa(\bar{\tau}) - 1 & (\text{if } i_1 \leq i \text{ and } i_1 - 1 < r < i) \\ \kappa(\bar{\tau}) & (\text{if otherwise}). \end{cases} \quad (26)$$

From (25), the blowups $\pi_r : X_r \rightarrow Y_r$ lift $f_r : Y_{r-1} \rightarrow Y_r$ to a biholomorphism $F_r : X_{r-1} \rightarrow X_r$:

$$\begin{array}{ccc} X_r & \xrightarrow{F_{r+1}} & X_{r+1} \\ \pi_r \downarrow & & \downarrow \pi_{r+1} \\ Y_r & \xrightarrow{f_{r+1}} & Y_{r+1}, \end{array}$$

and $\pi_\tau := \pi_n : X_\tau \rightarrow Y$ also lifts $f := f_n \circ \dots \circ f_1 : Y \rightarrow Y$ to the automorphism $F_\tau := F_n \circ \dots \circ F_1 : X_\tau \rightarrow X_\tau$, where $X_\tau := X_0 = X_n$ and $Y := Y_0 = Y_n$.

We now restrict our attention to the case where each component of $\bar{f} = (f_1, \dots, f_n)$ is a quadratic birational map with $Y_r = \mathbb{P}_r^2 = \mathbb{P}^2$ and τ is an original orbit data, and calculate the action of $F_\tau : X_\tau \rightarrow X_\tau$ on the cohomology group when \bar{f} is a realization of τ .

To this end, we recall some properties of quadratic maps. Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a quadratic birational map on \mathbb{P}^2 . It is known that f can be expressed as $f = \iota_- \circ \psi_\ell \circ \iota_+^{-1}$ for some $\ell \in \{1, 2, 3\}$, where $\iota_+, \iota_- : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ are linear transformations, and $\psi_\ell : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ are the quadratic birational maps given by

- (1) $\psi_1 : [x : y : z] \mapsto [yz : zx : xy]$ with $\psi_1^{-1} = \psi_1$,
- (2) $\psi_2 : [x : y : z] \mapsto [xz : yz : x^2]$ with $\psi_2^{-1} = \psi_2$,
- (3) $\psi_3 : [x : y : z] \mapsto [x^2 : xy : y^2 + xz]$ with $\psi_3^{-1} : [x : y : z] \mapsto [x^2 : xy : -y^2 + xz]$.

The indeterminacy sets of $\psi_\ell^{\pm 1}$ are expressed as $I(\psi_\ell^{\pm 1}) = \{p_{\ell,1}, p_{\ell,2}, p_{\ell,3}\}$, where

$$p_{1,1} = [1 : 0 : 0], \quad p_{2,2} > p_{2,1} = p_{1,2} = [0 : 1 : 0], \quad p_{3,3} > p_{3,2} > p_{3,1} = p_{2,3} = p_{1,3} = [0 : 0 : 1].$$

Then the geometry of the simple quadratic maps $\psi_\ell : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is described as follows. Let $\pi_\ell : X_\ell \rightarrow \mathbb{P}^2$ be the blowup of the cluster $\{p_{\ell,1}, p_{\ell,2}, p_{\ell,3}\}$, and let H_ℓ be the total transform of a line in \mathbb{P}^2 , $H_{\ell,1}$, $H_{\ell,2}$, $H_{\ell,3}$ be the strict transforms of the lines $x = 0$, $y = 0$, $z = 0$, respectively, and $E_{\ell,i}$ be the total transform of the point $p_{\ell,i}$ for $i = 1, 2, 3$. Then $E_{\ell,i}$ is linearly equivalent to $H_\ell - E_{\ell,j} - E_{\ell,k}$ for $\{i, j, k\} = \{1, 2, 3\}$. The birational map ψ_ℓ lifts to an automorphism $\tilde{\psi}_\ell : X_\ell \rightarrow X_\ell$, which sends irreducible rational curves $E_{\ell,i}$ to $H_{\ell,i}$ for $(\ell, i) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$, and sends irreducible rational curves $E_{\ell,j} - E_{\ell,k}$ to themselves for $(\ell, j, k) = (2, 1, 2), (3, 1, 2), (3, 2, 3)$ (see Figures 2-4). Moreover, ψ_ℓ sends a generic line to a conic passing through the three points $p_{\ell,1}, p_{\ell,2}, p_{\ell,3}$. Therefore, the action $\tilde{\psi}_\ell^* : H^2(X_\ell; \mathbb{Z}) \rightarrow H^2(X_\ell; \mathbb{Z})$ on the cohomology group $H^2(X_\ell; \mathbb{Z}) \cong \text{Pic}(X_\ell) = \mathbb{Z}[H_\ell] \oplus \mathbb{Z}[E_{\ell,1}] \oplus \mathbb{Z}[E_{\ell,2}] \oplus \mathbb{Z}[E_{\ell,3}]$ is given by

$$\tilde{\psi}_\ell^* : \begin{cases} [H_\ell] & \mapsto 2[H_\ell] - \sum_{i=1}^3 [E_{\ell,i}] \\ [E_{\ell,i}] & \mapsto [H_\ell] - [E_{\ell,j}] - [E_{\ell,k}] \quad (\{i, j, k\} = \{1, 2, 3\}). \end{cases} \quad (27)$$

Next, we consider a general quadratic birational map $f = \iota_- \circ \psi_\ell \circ \iota_+^{-1} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ with its inverse $f^{-1} = \iota_+ \circ \psi_\ell^{-1} \circ \iota_-^{-1} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$. Given $\{i, j, k\} = \{1, 2, 3\}$, put

$$p_i^\pm = \iota_\pm(p_{\ell,1}), \quad p_j^\pm = \iota_\pm(p_{\ell,2}), \quad p_k^\pm = \iota_\pm(p_{\ell,3}). \quad (28)$$

Then the indeterminacy points of $f^{\pm 1}$ are labeled as

$$I(f^{\pm 1}) = \{p_1^\pm, p_2^\pm, p_3^\pm\}. \quad (29)$$

Let $\pi^\pm : X^\pm \rightarrow \mathbb{P}^2$ be blowups of the clusters $\{p_1^\pm, p_2^\pm, p_3^\pm\}$, and let $H^\pm \subset X^\pm$ be the total transforms of a line in \mathbb{P}^2 under π^\pm , and $E_i^\pm \subset X^\pm$ be the exceptional divisors over the points p_i^\pm . Then the birational map $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ lifts to an isomorphism $\tilde{f} : X^+ \rightarrow X^-$. From (27), the cohomological action $\tilde{f}^* : H^2(X^-; \mathbb{Z}) \rightarrow H^2(X^+; \mathbb{Z})$ is given by

$$\tilde{f}^* : \begin{cases} [H^-] & \mapsto 2[H^+] - \sum_{i=1}^3 [E_i^+] \\ [E_i^-] & \mapsto [H^+] - [E_j^+] - [E_k^+] \quad (\{i, j, k\} = \{1, 2, 3\}). \end{cases} \quad (30)$$

Conversely, if a birational quadratic map $f = \iota_- \circ \psi_\ell \circ \iota_+^{-1} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ with the indeterminacy sets given in (29) lifts to $\tilde{f} : X^+ \rightarrow X^-$ satisfying (30), then the points p_i^\pm are expressed as (28) for some $\{i, j, k\} = \{1, 2, 3\}$. From here on, we assume that $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ lifts to $\tilde{f} : X^+ \rightarrow X^-$

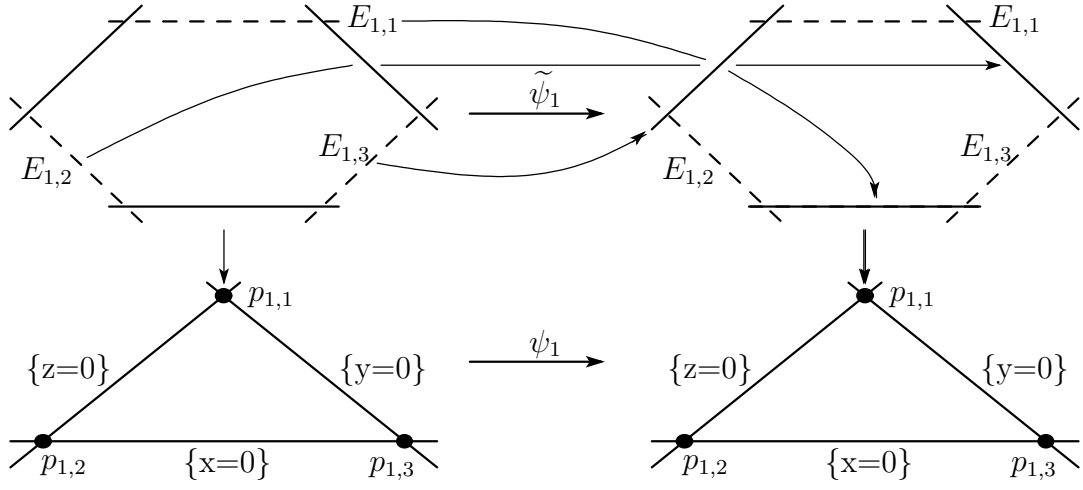


Figure 2: Geometry of $\psi_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$

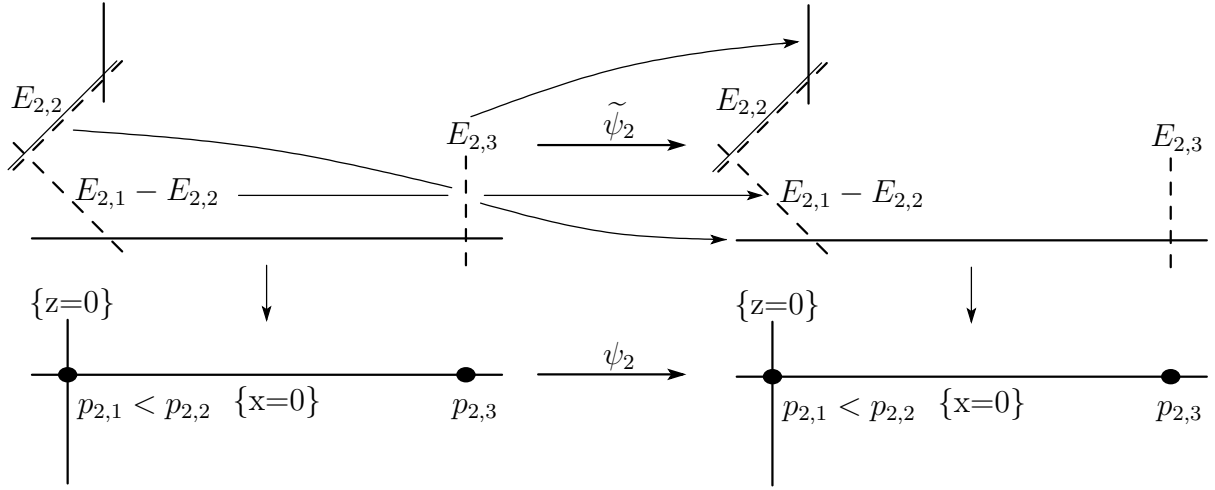


Figure 3: Geometry of $\psi_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$

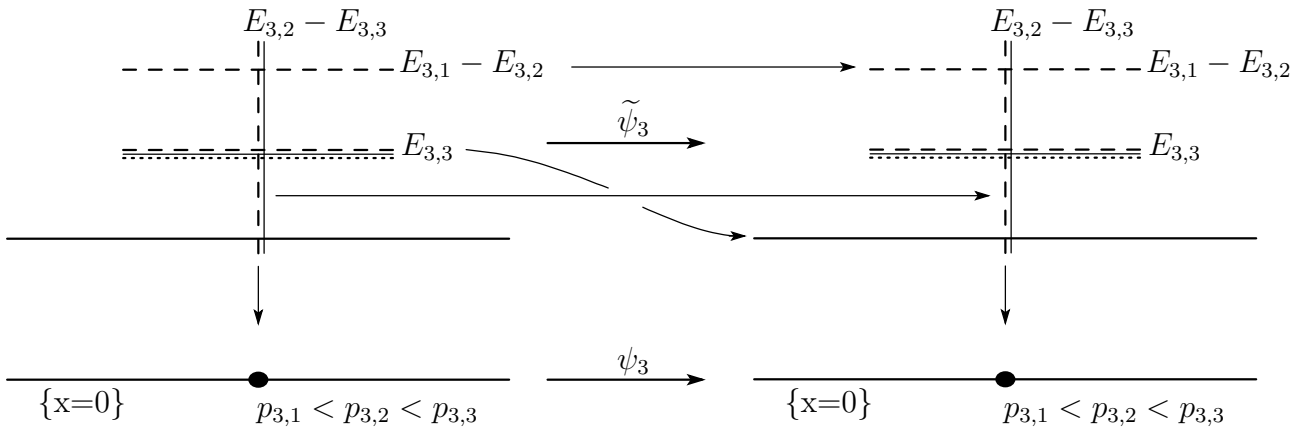


Figure 4: Geometry of $\psi_3 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$

satisfying (30). Then labeling $I(f) = \{p_1^+, p_2^+, p_3^+\}$ determines $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ and vice versa. In particular, it follows that $p_i^+ < p_j^+$ if and only if $p_i^- < p_j^-$.

Under these settings, we come back to consider an n -tuple $\bar{f} = (f_1, \dots, f_n)$ of quadratic birational maps $f_r : \mathbb{P}_{r-1}^2 \rightarrow \mathbb{P}_r^2$ with the indeterminacy sets $I(f_r^{\pm 1}) = \{p_{r,1}^{\pm}, p_{r,2}^{\pm}, p_{r,3}^{\pm}\}$. Since $\mathcal{K}(n) = \mathcal{K}(\bar{f}) = \mathcal{K}_+(\bar{f}) = \{\bar{\iota} = (i, \iota) \mid i = 1, 2, \dots, n, \iota = 1, 2, 3\}$, a generalized orbit data τ for \bar{f} becomes an original orbit data according to Definition 1.1. Moreover, \bar{f} can be called a realization of τ if the conditions in Definition 1.2 hold.

Remark 4.8 The value $\mu(\bar{\iota})$ stands for the length of the orbit segment $p_{\bar{\iota}}^0, p_{\bar{\iota}}^1, \dots, p_{\bar{\iota}}^{\mu(\bar{\iota})}$, while the value $\kappa(\bar{\iota})$ stands for the number of points in the orbit segment that lie on \mathbb{P}_n^2 . Moreover, the definition given in (7) of $\mu(\bar{\iota})$ yields a function $\mu : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\mu(\bar{\iota}) - i_1 + i + 1 \in n \cdot \mathbb{Z}$ for any $\bar{\iota} \in \mathcal{K}(n)$. Thus, there is one-to-one correspondence between the data (n, σ, κ) and (n, σ, μ) through equation (7). In what follows, we identify the orbit data $\tau = (n, \sigma, \kappa)$ with (n, σ, μ) , and write $\tau = (n, \sigma, \kappa) = (n, \sigma, \mu)$.

When \bar{f} is a realization of τ , the blowups $\pi_r : X_r \rightarrow Y_r = \mathbb{P}_r^2$ lift $f_r : \mathbb{P}_{r-1}^2 \rightarrow \mathbb{P}_r^2$ to bi-holomorphisms $F_r : X_{r-1} \rightarrow X_r$ and $\pi_r : X_r \rightarrow \mathbb{P}^2$ lifts $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ to the automorphism $F_\tau = F_n \circ \dots \circ F_1 : X_\tau \rightarrow X_\tau$. Let $H_r \subset X_r$ and $E_{\bar{\iota}, r}^k \subset X_r$ be the total transforms of a line in \mathbb{P}_r^2 and the point $p_{\bar{\iota}}^m$ with $k \geq 0$ and

$$m = \begin{cases} \theta_{i,r}(k) & (i \leq r) \\ \theta_{i,r}(k+1) & (i > r). \end{cases}$$

Then the cohomology group of X_r is $H^2(X_r; \mathbb{Z}) = \mathbb{Z}[H_r] \oplus (\oplus_{\bar{\iota} \in \mathcal{K}(n)} \oplus_{k=1}^{\kappa(r, \bar{\iota})} \mathbb{Z}[E_{\bar{\iota}, r}^{k-1}])$, where $\kappa(r, \bar{\iota})$ is given in (26). Moreover, from (30), the action $F_r^* : H^2(X_r; \mathbb{Z}) \rightarrow H^2(X_{r-1}; \mathbb{Z})$ is given by

$$F_r^* : \begin{cases} [H_r] & \mapsto 2[H_{r-1}] - \sum_{\ell=1}^3 [E_{\sigma^{-1}(r, \ell), r-1}^{\kappa(r-1, \sigma^{-1}(r, \ell)) - 1}] \\ [E_{(r, \ell_1), r}^0] & \mapsto [H_{r-1}] - [E_{\sigma^{-1}(r, \ell_2), r-1}^{\kappa(r-1, \sigma^{-1}(r, \ell_2)) - 1}] - [E_{\sigma^{-1}(r, \ell_3), r-1}^{\kappa(r-1, \sigma^{-1}(r, \ell_3)) - 1}] \quad (\{\ell_1, \ell_2, \ell_3\} = \{1, 2, 3\}) \\ [E_{\bar{\iota}, r}^k] & \mapsto [E_{\bar{\iota}, r-1}^k] \quad (i \neq r) \\ [E_{(r, l), r}^k] & \mapsto [E_{(r, l), r-1}^{k-1}]. \end{cases}$$

The composition $F_\tau^* = F_n^* \circ \dots \circ F_1^*$ acts on the cohomology group $H^2(X_\tau; \mathbb{Z}) = \mathbb{Z}[H] \oplus (\oplus_{\bar{\iota} \in \mathcal{K}(n)} \oplus_{k=1}^{\kappa(\bar{\iota})} \mathbb{Z}[E_{\bar{\iota}}^{k-1}])$, where $H := H_0 = H_n$ and $E_{\bar{\iota}}^k := E_{\bar{\iota}, 0}^k = E_{\bar{\iota}, n}^k$.

The above observation leads us to Definition 2.1. Namely, let $\phi_{\pi_\tau} : \mathbb{Z}^{1, N} \cong L_\tau \rightarrow H^2(X_\tau; \mathbb{Z})$ be the isomorphism defined by $\phi_{\pi_\tau}(e_0) = [H]$ and $\phi_{\pi_\tau}(e_{\bar{\iota}}^k) = E_{\bar{\iota}}^{k-1}$. Then it is easily seen that the automorphism $w_\tau : \mathbb{Z}^{1, N} \rightarrow \mathbb{Z}^{1, N}$ is realized by (π_τ, F_τ) , that is, $\phi_{\pi_\tau} \circ w_\tau = F_\tau^* \circ \phi_{\pi_\tau} : \mathbb{Z}^{1, N} \rightarrow H^2(X_\tau; \mathbb{Z})$. Summing up these discussions, we have the following proposition.

Proposition 4.9 *Assume that \bar{f} is a realization of τ . Then the blowup $\pi_\tau : X_\tau \rightarrow \mathbb{P}^2$ of $N = \sum_{\bar{\iota} \in \mathcal{K}(n)} \kappa(\bar{\iota})$ points $\{p_{\bar{\iota}}^m \mid \bar{\iota} \in \mathcal{K}(n), m = \theta_{i,0}(k), 1 \leq k \leq \kappa(\bar{\iota})\}$ lifts $f = f_n \circ \dots \circ f_1$ to the automorphism $F_\tau : X_\tau \rightarrow X_\tau$. Moreover, (π_τ, F_τ) realizes w_τ and F_τ has positive entropy $h_{\text{top}}(F_\tau) = \log \lambda(w_\tau) > 0$.*

5 Tentative Realizability

By restricting our attention to quadratic birational maps preserving a cuspidal cubic, we define a concept of tentative realization of orbit data. As is mentioned below, when such a realization exists, it is uniquely determined in some sense by the orbit data τ . From the characterization of composition of quadratic birational maps mentioned in Proposition 5.4, the existence of a tentative realization is investigated under the condition

$$\Gamma_\tau^{(1)} \cap P(\tau) = \emptyset, \quad (31)$$

where $\Gamma_\tau^{(1)}$ is given in (9), and $P(\tau)$ is the set of periodic roots with period ℓ_τ , that is,

$$P(\tau) := \{\alpha \in \Phi_N \mid w_\tau^{\ell_\tau}(\alpha) = \alpha\}. \quad (32)$$

First, we introduce some terminology used below. Let X be a smooth surface, C be a curve in X , and x be a proper point of the smooth locus C^* of C . Moreover, put $(X_0, C_0^*, x_0) := (X, C^*, x)$, and for $m > 0$, inductively determine (X_m, C_m^*, x_m) from the blowup $\pi_m : X_m \rightarrow X_{m-1}$ of $x_{m-1} \in C_{m-1}^*$, the strict transform C_m^* of C_{m-1}^* under π_m , and a unique point $x_m \in C_m^* \cap E_m$, where E_m stands for the exceptional curve of π_m . Then, x_m is called the *point in the m -th infinitesimal neighbourhood of x on C^** , or an *infinitely near point on C^** . Thus, a point in the m -th infinitesimal neighbourhood of x on C^* is uniquely determined. Moreover, if a cluster I consists of proper or infinitely near points on C^* , then we say that I is a *cluster in C^** .

Now let C be a cubic curve on \mathbb{P}^2 with a cusp singularity. In what follows, a coordinate on \mathbb{P}^2 is chosen so that $C = \{[x : y : z] \in \mathbb{P}^2 \mid yz^2 = x^3\} \subset \mathbb{P}^2$ with a cusp $[0 : 1 : 0]$. Then the smooth locus $C^* = C \setminus \{[0 : 1 : 0]\}$ is parametrized as $\mathbb{C} \ni t \mapsto [t : t^3 : 1] \in C^*$. We denote by $\mathcal{B}(C)$ the set of birational self-maps f of \mathbb{P}^2 such that $f(C) := \overline{f(C \setminus I(f))} = C$ and $[0 : 1 : 0] \notin I(f^{\pm 1})$, and denote by $\mathcal{Q}(C) \subset \mathcal{B}(C)$ and $\mathcal{L}(C) \subset \mathcal{B}(C)$ the subsets consisting of the quadratic maps in $\mathcal{B}(C)$ and of the linear maps in $\mathcal{B}(C)$, respectively. Any map $f \in \mathcal{B}(C)$ restricted to C^* is an automorphism of C^* expressed as

$$f|_{C^*} : C^* \ni [t : t^3 : 1] \mapsto [\delta(f) \cdot t + k_f : (\delta(f) \cdot t + k_f)^3 : 1] \in C^*, \quad (33)$$

for some $\delta(f) \in \mathbb{C}^\times$ and $k_f \in \mathbb{C}$. The value $\delta(f)$ is called the *determinant* of f . It is independent of the choice of the coordinate. Moreover, when $f \in \mathcal{Q}(C)$, it turns out that the indeterminacy sets $I(f^{\pm 1})$ are clusters in C^* (see Lemma 5.2).

We give the following definition for an n -tuple $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ of quadratic birational maps f_i preserving C .

Definition 5.1 An n -tuple $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ is called a *tentative realization* of an orbit data $\tau = (n, \sigma, \mu)$ if $p_\tau^{\mu(\bar{\tau})} \approx p_{\sigma(\bar{\tau})}^+$ for any $\bar{\tau} \in \mathcal{K}(n)$, where p_τ^m is given in (23) with f_r restricted to C and thus is well-defined.

We should note that a realization \bar{f} of τ is of course a tentative realization of τ , and thus the existence of a tentative realization is of interest to us.

Now, we describe a quadratic birational map $f \in \mathcal{Q}(C)$ in terms of the behavior of $f|_{C^*}$. The following proposition states that the configuration of $I(f^{-1})$ on C^* and the determinant $\delta(f)$ of f determine the map $f \in \mathcal{Q}(C)$ uniquely (see [5]).

Lemma 5.2 *A birational map f belongs to $\mathcal{Q}(C)$ if and only if there exist $d \in \mathbb{C}^\times$ and $b = (b_i)_{i=1}^3 \in \mathbb{C}^3$ with $b_1 + b_2 + b_3 \neq 0$ such that f can be expressed as $f = f_{d,b}$, where $f_{d,b} \in \mathcal{Q}(C)$ is a unique map determined by the following properties.*

$$(1) \quad \delta(f_{d,b}) = d.$$

$$(2) \quad p_i^- \approx [b_i : b_i^3 : 1] \in C^* \text{ for a suitable labeling } I(f_{d,b}^{-1}) = \{p_1^-, p_2^-, p_3^-\}.$$

Moreover, the map $f_{d,b} \in \mathcal{Q}(C)$ satisfies the following.

$$(1) \quad p_i^+ \approx [a_i : a_i^3 : 1] \in C^* \text{ for } I(f_{d,b}) = \{p_1^+, p_2^+, p_3^+\}, \text{ where } a_i := \frac{1}{d} \left\{ b_i - \frac{2}{3}(b_1 + b_2 + b_3) \right\}.$$

$$(2) \quad k_{f_{d,b}} = -\frac{1}{3}(b_1 + b_2 + b_3) \in \mathbb{C}^\times, \text{ where } k_f \text{ is given in (33).}$$

In a similar manner, any linear map $f \in \mathcal{L}(C)$ is determined uniquely by the determinant $\delta(f)$ of f (see [5]).

Lemma 5.3 *For any $d \in \mathbb{C}^\times$, there is a unique linear map $f \in \mathcal{L}(C)$ such that $\delta(f) = d$. In particular, the map $f \in \mathcal{L}(C)$ with $\delta(f) = 1$ is the identity. Moreover, for any $f \in \mathcal{L}(C)$, the automorphism $f|_{C^*}$ restricted to C^* is given by*

$$f|_{C^*} : [t : t^3 : 1] \mapsto [\delta(f) \cdot t : (\delta(f) \cdot t)^3 : 1].$$

Next, let us consider the composition $f = f_n \circ f_{n-1} \circ \cdots \circ f_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of quadratic birational maps $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$. A labeling $I(f_i^{-1}) = \{p_{i,1}^-, p_{i,2}^-, p_{i,3}^-\}$ determines $I(f_i) = \{p_{i,1}^+, p_{i,2}^+, p_{i,3}^+\}$ and the points:

$$\check{p}_{i,\iota}^+ := f_1^{-1}|_C \circ \cdots \circ f_{i-1}^{-1}|_C(p_{i,\iota}^+), \quad \check{p}_{i,\iota}^- := f_n|_C \circ \cdots \circ f_{i+1}|_C(p_{i,\iota}^-) \quad (34)$$

(see Figure 5). Then it is easy to see that $I(f^{\pm 1}) \subset \{\check{p}_{i,\iota}^\pm \mid (i, \iota) \in \mathcal{K}(n)\}$. Moreover, let $\delta(\bar{f})$ be the determinant of \bar{f} defined by $\delta(\bar{f}) = \prod_{j=1}^n \delta(f_j)$ or, in other words, $\delta(\bar{f}) = \delta(f)$.

Proposition 5.4 *Let $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ be an n -tuple of quadratic birational maps in $\mathcal{Q}(C)$ with $d = \delta(\bar{f}) \neq 1$, and let $\check{p}_{i,\iota}^\pm$ be the points given in (34) for a labeling $I(f_i^{-1}) = \{p_{i,1}^-, p_{i,2}^-, p_{i,3}^-\}$. Then there is a unique pair (v, s) of values $v = (v_\tau)_{\tau \in \mathcal{K}(n)} \in \mathbb{C}^{3n}$ and $s = (s_i)_{i=1}^n \in (\mathbb{C}^\times)^n$ such that (d, v, s) satisfies equation (13) and the composition $f = f_1 \circ \cdots \circ f_n$ satisfies*

$$(1) \quad f|_{C^*} : C^* \ni [t + \frac{1}{3}k(s) : (t + \frac{1}{3}k(s))^3 : 1] \mapsto [d \cdot t + \frac{1}{3}k(s) : (d \cdot t + \frac{1}{3}k(s))^3 : 1] \in C^*, \text{ where } k(s) \text{ is given in (15),}$$

$$(2) \quad \check{p}_{i,\iota}^- \approx [v_{i,\iota} + \frac{1}{3}k(s) : (v_{i,\iota} + \frac{1}{3}k(s))^3 : 1] \in C^*,$$

(3) $\check{p}_{i,\iota}^+ \approx [u_{i,\iota} + \frac{1}{3}k(s) : (u_{i,\iota} + \frac{1}{3}k(s))^3 : 1] \in C^*$, where

$$u_{i,\iota} := \frac{1}{d} \{v_{i,\iota} - (d-1) \cdot s_i\}. \quad (35)$$

Conversely, for any $d \in \mathbb{C} \setminus \{0, 1\}$, $v \in \mathbb{C}^{3n}$ and $s \in (\mathbb{C}^\times)^n$ satisfying equation (13), there exists an n -tuple $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ such that, for a suitable labeling $I(f_i^{-1}) = \{p_{i,1}^-, p_{i,2}^-, p_{i,3}^-\}$, the composition $f = f_1 \circ \dots \circ f_n$ satisfies conditions (1)–(3). Moreover, the map \bar{f} is determined uniquely by (d, v, s) in the sense that if $\bar{f} = (f_1, \dots, f_n)$ and $\bar{f}' = (f'_1, \dots, f'_n)$ are determined by (d, v, s) , then there are linear maps $g_1, \dots, g_{n-1} \in \mathcal{L}(C)$ such that $f_j = g_{j-1} \circ f'_j \circ g_j$ for any $j = 1, \dots, n$, where $g_0 = g_n = \text{id}$.

Proof. From Lemma 5.2, each map $f_i \in \mathcal{Q}(C)$ is given by $f_i = f_{d_i, (b_{i,\iota})}$ for some $d_i \in \mathbb{C}^\times$ and $(b_{i,\iota})_{\iota=1}^3 \in \mathbb{C}^3$ with $b_i := b_{i,1} + b_{i,2} + b_{i,3} \neq 0$. Then the maps f_i and f restricted to C^* can be expressed as $f_i|_{C^*}([t : t^3 : 1]) = [y_i(t) : y_i(t)^3 : 1]$ and $f|_{C^*}([t : t^3 : 1]) = [y(t) : y(t)^3 : 1]$, respectively, where $y_i, y : \mathbb{C} \rightarrow \mathbb{C}$ are the maps given by

$$y_i(t) = d_i \cdot t - \frac{1}{3}b_i,$$

and $y := y_n \circ y_{n-1} \circ \dots \circ y_1$. Now we put $\check{d}_i := d_{i+1} \cdot d_{i+2} \cdot \dots \cdot d_n$, $a_{i,\iota} := (b_{i,\iota} - \frac{2}{3}b_i)/d_i$ and

$$\check{a}_{i,\iota} := y_1^{-1} \circ \dots \circ y_{i-1}^{-1}(a_{i,\iota}), \quad \check{b}_{i,\iota} := y_n \circ \dots \circ y_{i+1}(b_{i,\iota}), \quad \check{b}_i := \check{b}_{i,1} + \check{b}_{i,2} + \check{b}_{i,3}.$$

Then it follows that $\check{p}_{i,\iota}^+ \approx [\check{a}_{i,\iota} : \check{a}_{i,\iota}^3 : 1]$, $\check{p}_{i,\iota}^- \approx [\check{b}_{i,\iota} : \check{b}_{i,\iota}^3 : 1]$ and $d = \check{d}_0$. A little calculation shows that

$$\begin{aligned} y(t) &= d \cdot t - \frac{1}{3} \sum_{r=1}^n 2^{r-1} \cdot \check{b}_r, \\ \check{b}_{i,\iota} &= \check{d}_i \cdot b_{i,\iota} - \frac{1}{3} \sum_{r=i+1}^n \check{d}_r \cdot b_r = \check{d}_i \cdot b_{i,\iota} - \frac{d-1}{3} \sum_{r=i+1}^n s_r, \\ \check{a}_{i,\iota} &= \frac{1}{d} \left(\check{d}_i \cdot b_{i,\iota} - \check{d}_i \cdot b_i + \frac{1}{3} \sum_{r=1}^i \check{d}_r \cdot b_r \right) = \frac{1}{d} \left\{ \check{d}_i \cdot b_{i,\iota} - (d-1) \cdot s_i + \frac{d-1}{3} \sum_{r=1}^i s_r \right\}, \end{aligned}$$

where $s_i := \check{d}_i \cdot b_i / (d-1) \neq 0$. If we put

$$v_{i,\iota} := \check{d}_i \cdot b_{i,\iota} - \frac{1}{3} \left(\sum_{r=1}^i s_r + d \cdot \sum_{r=i+1}^n s_r \right),$$

then we have

$$v_{i,1} + v_{i,2} + v_{i,3} = \check{d}_i \cdot b_i - \left(\sum_{r=1}^i s_r + d \cdot \sum_{r=i+1}^n s_r \right) = - \sum_{r=1}^{i-1} s_r + (d-2)s_i - d \sum_{r=i+1}^n s_r,$$

which shows that equation (13) holds. Moreover, since $\check{b}_i = \check{d}_i \cdot b_i - (d-1) \cdot \sum_{r=i+1}^n s_r = (d-1) \cdot \{s_i - \sum_{r=i+1}^n s_r\}$ and thus $\sum_{r=1}^n 2^{r-1} \cdot \check{b}_r = (d-1) \cdot k(s)$, the map $y(t) = d \cdot t - (d-1) \cdot k(s)/3$

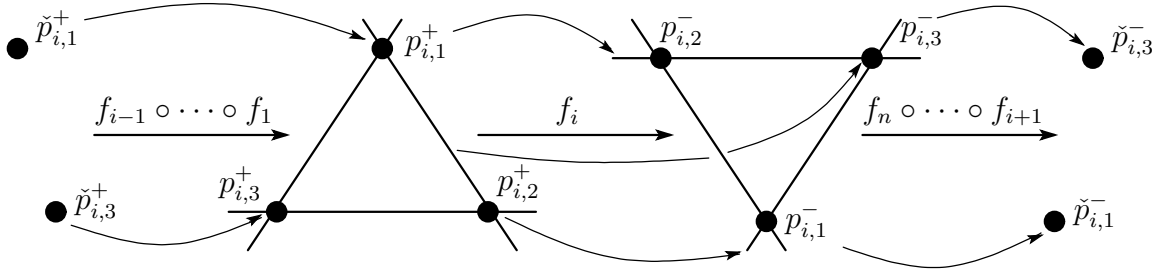


Figure 5: The points $\check{p}_{i,\ell}^+ \in I(f)$ and $\check{p}_{i,\ell}^- \in I(f^{-1})$

has the unique fixed point $k(s)/3$ under the assumption that $d \neq 1$. Finally, we have

$$\begin{aligned}
\check{b}_{i,\ell} &= \check{d}_i \cdot b_{i,\ell} - \frac{d-1}{3} \sum_{r=i+1}^n s_r = v_{i,\ell} + \frac{1}{3} \left(\sum_{r=1}^i s_r + d \cdot \sum_{r=i+1}^n s_r \right) - \frac{d-1}{3} \sum_{r=i+1}^n s_r = v_{i,\ell} + \frac{1}{3} k(s), \\
d \cdot \check{a}_{i,\ell} &= \check{d}_i \cdot b_{i,\ell} - (d-1) \cdot s_i + \frac{d-1}{3} \sum_{r=1}^i s_r \\
&= v_{i,\ell} + \frac{1}{3} \left(\sum_{r=1}^i s_r + d \cdot \sum_{r=i+1}^n s_r \right) - (d-1) \cdot s_i + \frac{d-1}{3} \sum_{r=1}^i s_r \\
&= v_{i,\ell} - (d-1) \cdot s_i + \frac{d}{3} k(s).
\end{aligned}$$

Thus conditions (1)–(3) hold.

Conversely, for any $d \neq 1$, (s_i) and $(v_{i,\ell})$ satisfying (13), the maps $(f_i) = (f_{d_i, (b_{i,\ell})})$ with

$$\begin{aligned}
d_i &= \begin{cases} 1, & (i \neq n) \\ d, & (i = n) \end{cases} \\
b_{i,\ell} &= \frac{1}{d} \left\{ v_{i,\ell} + \frac{1}{3} \left(\sum_{r=1}^i s_r + d \cdot \sum_{r=i+1}^n s_r \right) \right\}
\end{aligned}$$

give the birational map $f = f_n \circ \dots \circ f_1$ satisfying conditions (1)–(3). Moreover, assume that there are two n -tuples $\bar{f} = (f_1, \dots, f_n)$ and $\bar{f}' = (f'_1, \dots, f'_n)$ in $\mathcal{Q}(C)^n$ such that $f = f_n \circ \dots \circ f_1$ and $f' = f'_n \circ \dots \circ f'_1$ satisfy conditions (1)–(3) for (d, v, s) . Put $g_j := f'_{j+1} \circ \dots \circ f'_n \circ f_n^{-1} \circ \dots \circ f'_{j+1} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$. Then one has $f_j = g_{j-1} \circ f'_j \circ g_j$, where $g_n = \text{id}$. It follows from condition (2) that g_j is a linear map in $\mathcal{L}(C)$, and from condition (1) that the determinant of g_1 is given by $\delta(g_1) = \delta(f') \cdot \delta(f)^{-1} = 1$, which means that $g_1 = \text{id}$ (see Lemma 5.3). This completes the proof. \square

Corollary 5.5 *Let τ be an orbit data with $\lambda(w_\tau) > 1$, d be a root of $S_{w_\tau}(t) = 0$ and $s \neq 0$ be a unique solution of equation (21) (see Corollary 3.4). Then s satisfies $s_j \neq 0$ for any $1 \leq j \leq n$ if and only if there is a tentative realization \bar{f} of τ with $\delta(\bar{f}) = d$. Moreover, the tentative realization \bar{f} of τ is uniquely determined in the sense that if there are two tentative realizations $\bar{f} = (f_1, \dots, f_n)$ and $\bar{f}' = (f'_1, \dots, f'_n)$ of τ with $\delta(\bar{f}) = \delta(\bar{f}') = d$, then there are linear maps $g_1, \dots, g_n \in \mathcal{L}(C)$ such that $f_j = g_{j-1} \circ f'_j \circ g_j$ for any $j = 1, \dots, n$, where $g_0 := g_n$.*

Proof. First, assume that there is a tentative realization \bar{f} of τ with $\delta(\bar{f}) = d$. Then we notice that it is unique. Indeed, \bar{f} satisfies $p_{\bar{\tau}}^{\mu(\bar{\tau})} \approx p_{\sigma(\bar{\tau})}^+$, or equivalently, $f|_C^{\kappa(\bar{\tau})-1}(\bar{p}_{\bar{\tau}}^-) \approx \check{p}_{\sigma(\bar{\tau})}^+$ and thus $d^{\kappa(\bar{\tau})-1} \cdot v_{\bar{\tau}} = u_{\sigma(\bar{\tau})}$ for any $\bar{\tau} \in \mathcal{K}(n)$. Therefore, from (35), the pair $(v, s) \in \mathbb{C}^{3n} \times (\mathbb{C}^\times)^n$ given in Proposition 5.4 satisfies (13) and (14). Since a solution of (13) and (14) is unique, up to a constant multiple (see Corollary 3.4), the map f is unique, up to conjugacy by a linear map in $\mathcal{L}(C)$, and so is \bar{f} (see Lemma 5.3). Moreover, s satisfies $s_j \neq 0$ for any $1 \leq j \leq n$.

Conversely, assume that s satisfies $s_j \neq 0$ for any $1 \leq j \leq n$. From Corollary 3.4, there is a solution (v, s) of (13) and (14). Hence, Proposition 5.4 gives an n -tuple $\bar{f} = (f_1, \dots, f_n) \in \mathcal{Q}(C)^n$ such that $f = f_n \circ \dots \circ f_1$ satisfies conditions (1)–(3) in Proposition 5.4 and thus $\delta(\bar{f}) = d$. In view of (14) and (35), one has $d^{\kappa(\bar{\tau})-1} \cdot v_{\bar{\tau}} = u_{\sigma(\bar{\tau})}$ and so $p_{\bar{\tau}}^{\mu(\bar{\tau})} \approx p_{\sigma(\bar{\tau})}^+$ for any $\bar{\tau} \in \mathcal{K}(n)$, which means that \bar{f} is the tentative realization of τ with $\delta(\bar{f}) = d$. \square

Now we fix an orbit data τ with $\lambda(w_\tau) > 1$. Then from Corollary 3.4, $\lambda(w_\tau)$ is a root of $\chi_\tau(t) = 0$ and there is a unique solution $s_\tau \neq 0 \in \mathbb{C}^n$ of the equation (21) with $d = \lambda(w_\tau)$.

Lemma 5.6 *For each $1 \leq j \leq n$, α_j^c belongs to $P(\tau)$ if and only if $(s_\tau)_j = 0$, where α_j^c is given in (11) and $(s_\tau)_j$ is the j -th component of s_τ .*

Proof. Assume that $\alpha_j^c \in P(\tau)$, which is equivalent to saying that $(\alpha_j^c, \bar{v}_\delta) = 0$ from Lemma 3.1, where $\delta := \lambda(w_\tau)$. By (16), we have

$$\begin{aligned} (\alpha_j^c, \bar{v}_\delta) &= (e_0 - e_{(j,1)\tau}^1 - e_{(j,2)\tau}^1 - e_{(j,3)\tau}^1, q_{j+1} \circ \dots \circ q_n(\bar{v}_\delta)) \\ &= \left(\sum_{m=1}^j (s_\tau)_m + \delta \sum_{m=j+1}^n (s_\tau)_m \right) + \sum_{\iota=1}^3 v_{j,\iota} \\ &= \left(\sum_{m=1}^j (s_\tau)_m + \delta \sum_{m=j+1}^n (s_\tau)_m \right) + \left(- \sum_{m=1}^{j-1} (s_\tau)_m + (\delta - 2)(s_\tau)_j - \delta \sum_{m=j+1}^n (s_\tau)_m \right) \\ &= (\delta - 1)(s_\tau)_j. \end{aligned}$$

Thus the equation $(\alpha_j^c, \bar{v}_\delta) = 0$ is equivalent to saying that $(s_\tau)_j = 0$, since $\delta > 1$. \square

Propositions 5.7, 5.8 and 5.9 mentioned below run parallel with Theorems 1.3–1.5. Namely, Proposition 5.7 states that there is a tentative realization of τ under condition (31), Proposition 5.8 states that the sibling $\tilde{\tau}$ of any orbit data satisfies condition (31), and finally Proposition 5.9 gives a sufficient condition for (31).

Proposition 5.7 *Assume that an orbit data τ satisfies $\lambda(w_\tau) > 1$ and condition (31). Then there is a unique tentative realization \bar{f} of τ such that $\delta(\bar{f}) = \lambda(w_\tau) > 1$. Conversely, if there is a tentative realization \bar{f} of τ such that $\delta(\bar{f}) = \lambda(w_\tau) > 1$, then τ satisfies condition (31).*

Proof. This proposition follows easily from Corollary 5.5 and Lemma 5.6. \square

Proposition 5.8 *There is a data $\tilde{\tau} = (\tilde{n}, \tilde{\sigma}, \tilde{\kappa})$ with $\tilde{n} \leq n$ such that $\delta = \lambda(w_\tau) = \lambda(w_{\tilde{\tau}})$ and $(s_{\tilde{\tau}})_j \neq 0$ for any $1 \leq j \leq \tilde{n}$. In particular, $\tilde{\tau}$ satisfies condition (31).*

Proof. Let $(v, s_\tau) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be the unique solution of (13) and (14) as in Corollary 3.4, and assume that $(s_\tau)_j = 0$. Then we put $\tilde{n} := n - 1$, and for any $\bar{\iota} = (i, \iota) \in \mathcal{K}(\tilde{n}) \cong \{(i, \iota) \in \mathcal{K}(n) \mid i \neq j\}$, choose $k(\bar{\iota})$ so that $i_1 = \dots = i_{k(\bar{\iota})-1} = j$ but $i_{k(\bar{\iota})} \neq j$. The new orbit data $\tilde{\tau} = (\tilde{n}, \tilde{\sigma}, \tilde{\kappa})$ is defined by $\tilde{\sigma}(\bar{\iota}) := \sigma^{k(\bar{\iota})}(\bar{\iota})$ and $\tilde{\kappa}(\bar{\iota}) := \sum_{k=0}^{k(\bar{\iota})-1} \kappa(\sigma^k(\bar{\iota}))$ for any $\bar{\iota} \in \mathcal{K}(\tilde{n})$. Then, since $v_{\sigma(\bar{\iota})} = \delta^{\kappa(\bar{\iota})-1} \cdot v_{\bar{\iota}} + (\delta - 1) \cdot (s_\tau)_{i_1}$ and $(s_\tau)_j = 0$, we have $v_{\tilde{\sigma}(\bar{\iota})} = \delta^{\tilde{\kappa}(\bar{\iota})-1} \cdot v_{\bar{\iota}} + (\delta - 1) \cdot (s_\tau)_{i_1}$ for any $\bar{\iota} \in \mathcal{K}(\tilde{n})$, where $\bar{\iota}_m^\sigma = (i_m^\sigma, \iota_m^\sigma) := \tilde{\sigma}^m(\bar{\iota})$. Moreover, as v satisfies (13) with $d = \delta$ and with $s = s_\tau$, $(v_{\bar{\iota}})_{\bar{\iota} \in \mathcal{K}(\tilde{n})}$ satisfies (13) with $n = \tilde{n}$, $d = \delta$ and $s = s_{\tilde{\tau}} = ((s_\tau)_1, \dots, (s_\tau)_{j-1}, (s_\tau)_{j+1}, \dots, (s_\tau)_n) \neq 0$. Hence, we have $\mathcal{A}_{\tilde{\tau}}(\delta) s_{\tilde{\tau}} = 0$ and $\delta = \lambda(w_{\tilde{\tau}}) = \lambda(w_\tau)$.

Therefore, either $(s_{\tilde{\tau}})_{\hat{j}} \neq 0$ for any \hat{j} , or we can repeat the above argument to eliminate $(s_{\tilde{\tau}})_{\hat{j}} = 0$ from $s_{\tilde{\tau}}$. Since each step reduces n by 1, $\tilde{\tau}$ satisfies $(s_{\tilde{\tau}})_{\hat{j}} \neq 0$ for any \hat{j} after finitely many steps. \square

Proposition 5.9 *For any orbit data τ satisfying conditions (1) and (2) in Theorem 1.5, there is a real number δ with $2^n - 1 < \delta < 2^n$ such that $\chi_\tau(\delta) = 0$, and thus $\lambda(w_\tau) = \delta > 1$. Moreover, τ satisfies condition (31).*

The proof of this proposition is given in Section 7.

Remark 5.10 As is mentioned in Proposition 5.7, the tentative realization \bar{f} of τ with $\delta(\bar{f}) = \lambda(w_\tau)$ is unique. However, when $p_{\bar{\iota}}^- \approx p_{\bar{\iota}'}^-$ for some $\bar{\iota} \neq \bar{\iota}' \in \mathcal{K}(n)_i = \{(i, 1), (i, 2), (i, 3)\}$, there remains an ambiguity about how to label the indeterminacy points. Namely, either $p_{\bar{\iota}}^\pm < p_{\bar{\iota}'}^\pm$ or $p_{\bar{\iota}}^\pm > p_{\bar{\iota}'}^\pm$ holds. In this case, for a fixed n -tuple $(\prec_i) \in \mathcal{T}(\tau)$ of total orders (see Definition 2.3), we choose the labeling so that if $p_{\bar{\iota}}^- \approx p_{\bar{\iota}'}^-$ and $\bar{\iota}' \prec_i \bar{\iota}$, then $p_{\bar{\iota}'}^- < p_{\bar{\iota}}^-$ and $p_{\bar{\iota}'}^+ < p_{\bar{\iota}}^+$.

6 Realizability

Under condition (31), we study the tentative realization \bar{f} of τ given in Proposition 5.7, and show that \bar{f} becomes a realization of τ when τ satisfies the condition

$$\Gamma_\tau^{(2)} \cap P(\tau) = \emptyset, \quad (36)$$

where $\Gamma_\tau^{(2)}$ and $P(\tau)$ are given in (10) and (32), respectively. In the last part of this section, the main theorems of this paper are established.

First, we prove the following lemma.

Lemma 6.1 *Assume that τ satisfies condition (31). Then, for the tentative realization \bar{f} of τ mentioned in Proposition 5.7, the following hold.*

- (1) $p_{\bar{\iota}}^m \approx p_{\bar{\iota}'}^-$ if and only if $\alpha_{\bar{\iota}, \bar{\iota}'}^k \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$, where $m = \theta_{i, i'}(k) \geq 0$.
- (2) $p_{\bar{\iota}}^m \approx p_{\bar{\iota}'}^+$ with $\mu(\bar{\iota}') \leq m$ if and only if $\alpha_{\bar{\iota}, \bar{\iota}'}^k \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$, where k is determined by $\theta_{i, i'}(k) + \mu(\bar{\iota}') = m$.
- $p_{\bar{\iota}}^m \approx p_{\bar{\iota}'}^+$ with $m \leq \mu(\bar{\iota}')$ if and only if $\alpha_{\bar{\iota}, \bar{\iota}'}^k \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$, where k is determined by $\theta_{i', i}(k) + m = \mu(\bar{\iota}')$.

- (3) $p_{\bar{\tau}}^- \approx p_{\bar{\tau}'}^-$ if and only if $\alpha_{\bar{\tau}, \bar{\tau}'}^0 \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$ with $i = i'$.
- (4) $p_{\bar{\tau}_1}^+ \approx p_{\bar{\tau}'_1}^+$ with $\mu(\bar{\tau}) \geq \mu(\bar{\tau}')$ if and only if $\alpha_{\bar{\tau}, \bar{\tau}'}^k \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$, where k is determined by $\theta_{i, i'}(k) + \mu(\bar{\tau}') = \mu(\bar{\tau})$.

Proof. We only prove assertion (1) as the remaining statements can be treated in a similar manner. Assume that $\alpha_{\bar{\tau}, \bar{\tau}'}^k \in P(\tau)$, which is equivalent to saying that $(\alpha_{\bar{\tau}, \bar{\tau}'}^k, \bar{v}_\delta) = 0$ by Lemma 3.1. From (16), this means that

$$0 = (\alpha_{\bar{\tau}, \bar{\tau}'}^k, \bar{v}_\delta) = (e_{\bar{\tau}_\tau}^{k+1} - e_{\bar{\tau}'_\tau}^1, q_{i'+1} \circ \cdots \circ q_n(\bar{v}_\delta)) = \delta^k \cdot v_{\bar{\tau}} - v_{\bar{\tau}'}, \quad (37)$$

where the last equality follows from the fact that the coefficients of $e_{\bar{\tau}_\tau}^{k+1}$ and $e_{\bar{\tau}'_\tau}^1$ in $q_{i'+1} \circ \cdots \circ q_n(\bar{v}_\delta)$ are $\delta^k \cdot v_{\bar{\tau}}$ and $v_{\bar{\tau}'}$ respectively, since $\theta_{i, i'}(k) \geq 0$. Thus we have $\check{p}_{\bar{\tau}'}^- \approx f|_C^k(\check{p}_{\bar{\tau}}^-)$ and

$$\begin{aligned} p_{\bar{\tau}'}^- &= f_{i'+1}^{-1}|_C \circ \cdots \circ f_n^{-1}|_C(\check{p}_{\bar{\tau}'}^-) \approx f_{i'+1}^{-1}|_C \circ \cdots \circ f_n^{-1}|_C(f|_C^k(\check{p}_{\bar{\tau}}^-)) \\ &= f_{i'+1}^{-1}|_C \circ \cdots \circ f_n^{-1}|_C \circ (f|_C)^k \circ f_n|_C \circ \cdots \circ f_{i+1}|_C(p_{\bar{\tau}}^-) = p_{\bar{\tau}}^{-, \theta_{i, i'}(k)}. \end{aligned}$$

Conversely, if $p_{\bar{\tau}'}^- \approx p_{\bar{\tau}}^{-, \theta_{i, i'}(k)}$, then it follows from the above arguments that $\alpha_{\bar{\tau}, \bar{\tau}'}^k \in P(\tau)$. Therefore, assertion (1) of the lemma is established. \square

In order to see whether the tentative realization \bar{f} becomes a realization, we restate Lemma 4.7 as follows.

Proposition 6.2 *Assume that \bar{f} is a tentative realization of τ . Then \bar{f} is a realization of τ if and only if there is a total order \prec on $\mathcal{K}(n)$ such that the following conditions hold:*

- (1) *If $i = i'$ and $p_{\bar{\tau}'}^- < p_{\bar{\tau}}^-$, then $\bar{\tau}' \not\preceq \bar{\tau}$.*
- (2) *If $i_1 = i'_1$ and $p_{\bar{\tau}'_1}^+ < p_{\bar{\tau}_1}^+$, then $\bar{\tau}' \not\preceq \bar{\tau}$.*
- (3) *If $p_{\bar{\tau}}^m \approx p_{\bar{\tau}'}^-$ for $0 < m \leq \mu(\bar{\tau})$, then $\bar{\tau}' \not\preceq \bar{\tau}$.*
- (4) *If $p_{\bar{\tau}}^m \approx p_{\bar{\tau}'_1}^+$ for $0 \leq m < \mu(\bar{\tau})$, then $\bar{\tau}' \not\preceq \bar{\tau}$.*

Proof. Assume that there is a total order \prec on $\mathcal{K}(n)$ satisfying conditions (1)–(4). Under the notation of Lemma 4.7, consider the sequence $\bar{\tau}^{(1)} \prec \cdots \prec \bar{\tau}^{(3n)}$, and assume that $(p_{\bar{\tau}^{(\ell)}}^-, p_{\sigma(\bar{\tau}^{(\ell)})}^+)$ is a proper pair of $\bar{g}_{\ell-1} = (g_{\ell-1,1}, \dots, g_{\ell-1,n})$ with length $\mu(\bar{\tau}^{(\ell)})$ for any $\ell = 1, \dots, j-1$. Then from (25), we have

$$I(g_{j-1,r}^{-1}) = \{p_{\bar{\tau}}^- | \bar{\tau}^{(j)} \prec \bar{\tau}, i = r\}, \quad I(g_{j-1,r}) = \{p_{\sigma(\bar{\tau})}^+ | \bar{\tau}^{(j)} \prec \bar{\tau}, i_1 = r\}.$$

Thus, $p_{\bar{\tau}^{(j)}}^-$ and $p_{\sigma(\bar{\tau}^{(j)})}^+$ are proper points from conditions (1) and (2) respectively. Moreover, one has $p_{\bar{\tau}^{(j)}}^m \not\approx p_{\bar{\tau}'_1}^+$ for any $0 \leq m < \mu(\bar{\tau})$ and $\bar{\tau}' \not\preceq \bar{\tau}^{(j)}$, and $p_{\bar{\tau}^{(j)}}^m \not\approx p_{\bar{\tau}'}^-$ for any $0 < m \leq \mu(\bar{\tau})$ and $\bar{\tau}' \not\preceq \bar{\tau}^{(j)}$ from conditions (3) and (4), respectively. Since $p_{\bar{\tau}^{(j)}}^{\mu(\bar{\tau}^{(j)})} \approx p_{\sigma(\bar{\tau}^{(j)})}^+$ and $p_{\bar{\tau}^{(j)}}^{\mu(\bar{\tau}^{(j)})}$ is also a proper point, we have $p_{\bar{\tau}^{(j)}}^{\mu(\bar{\tau}^{(j)})} = p_{\sigma(\bar{\tau}^{(j)})}^+$. Therefore, $(p_{\bar{\tau}^{(j)}}^-, p_{\sigma(\bar{\tau}^{(j)})}^+)$ is a proper pair of \bar{g}_{j-1} with length $\mu(\bar{\tau}^{(j)})$, and \bar{f} is a realization of τ by Lemma 4.7.

Similarly, if \bar{f} is a realization of τ , it is easy to see that the total order \prec mentioned in Lemma 4.7 satisfies conditions (1)–(4), and so the proof is complete. \square

From the results mentioned above, we have the following three propositions, which also run parallel with Theorems 1.3–1.5 in a similar way to Propositions 5.7, 5.8 and 5.9.

Proposition 6.3 *Let τ be an orbit data satisfying $\lambda(w_\tau) > 1$ and condition (31), and \bar{f} be the tentative realization mentioned in Proposition 5.7. Then, τ satisfies condition (36) if and only if \bar{f} is a realization of τ .*

Proof. From Proposition 6.2, we will show that τ satisfies condition (36) if and only if there is a total order \prec on $\mathcal{K}(n)$ satisfying conditions (1)–(4) in Proposition 6.2.

First, we assume that τ satisfies condition (36). For a fixed $(\prec_i) \in \mathcal{T}(\tau)$ (see Definition 2.3), let $\hat{P}(\tau; (\prec_i))$ be the set of elements $\alpha_{\bar{t}, \bar{t}'}^k$ in $P(\tau)$ satisfying either $\theta_{i, i'}(k) = 0$ and $\bar{t}' \prec_i \bar{t}$ or $\theta_{i, i'}(k) > 0$. Moreover, we fix a total order \prec on $\mathcal{K}(n)$ such that if $\alpha_{\bar{t}, \bar{t}'}^k \in \hat{P}(\tau; (\prec_i))$, then $\bar{t}' \prec \bar{t}$. This total order is well-defined. To see this, we show that if $\alpha_{\bar{t}^{(\ell)}, \bar{t}^{(\ell+1)}}^{k_\ell} \in \hat{P}(\tau; (\prec_i))$ with $\bar{t}^{(1)} = \bar{t}^{(j+1)}$ for some j , then $\bar{t}^{(1)} = \bar{t}^{(2)} = \dots = \bar{t}^{(j)}$. Indeed, since $\alpha_{\bar{t}^{(\ell)}, \bar{t}^{(\ell+1)}}^{k_\ell} \notin \Gamma_\tau^{(2)}$, one has $0 \leq \theta_{i(\ell), i(\ell+1)}(k_\ell) + \mu(\bar{t}^{(\ell+1)}) \leq \mu(\bar{t}^{(\ell)})$, which yields $\sum_{\ell=1}^j \theta_{i(\ell), i(\ell+1)}(k_\ell) = 0$ and $\theta_{i(\ell), i(\ell+1)}(k_\ell) = 0$ for any ℓ . As $\alpha_{\bar{t}^{(\ell)}, \bar{t}^{(\ell+1)}}^{k_\ell} \in \hat{P}(\tau; (\prec_i))$ with $\theta_{i(\ell), i(\ell+1)}(k_\ell) = 0$, we have $\bar{t}^{(\ell+1)} \prec_{i(\ell)} \bar{t}^{(\ell)}$ for any ℓ , and $\bar{t}^{(1)} = \bar{t}^{(2)} = \dots = \bar{t}^{(j)}$. Then it is easy to see from Lemma 6.1 that the total order \prec satisfies conditions (1) and (3) in Proposition 6.2 (see Remark 5.10). Thus, we need to prove that this total order satisfies the remaining conditions.

To prove condition (2), assume that $i_1 = i'_1$ and $p_{\bar{t}'_1}^+ < p_{\bar{t}_1}^+$. Then we have $p_{\bar{t}'_1}^- < p_{\bar{t}_1}^-$ and thus $\bar{t}'_1 \prec_{i_1} \bar{t}_1$. It follows from assertion (4) of Lemma 6.1 that $\bar{\Gamma}_\tau^{(2)} \cap P(\tau)$ contains either $\alpha_{\bar{t}, \bar{t}'}^k$ with $\mu(\bar{t}') + \theta_{i, i'}(k) = \mu(\bar{t})$ and with $\theta_{i, i'}(k) \geq 0$, or $\alpha_{\bar{t}', \bar{t}}^k$ with $\mu(\bar{t}) + \theta_{i', i}(k) = \mu(\bar{t}')$ and with $\theta_{i', i}(k) > 0$. However, the latter case does not occur, since $\alpha_{\bar{t}', \bar{t}}^k \in \Gamma_\tau^{(2)}$. In the former case, if $\theta_{i, i'}(k) > 0$, then one has $\bar{t}' \prec \bar{t}$. Similarly, if $\theta_{i, i'}(k) = 0$ then we have $\bar{t}' \prec_i \bar{t}$, and thus $\bar{t}' \prec \bar{t}$, since $\bar{t}'_1 \prec_{i_1} \bar{t}_1$ and $\alpha_{\bar{t}, \bar{t}'}^k \notin \Gamma_\tau^{(2)}$. Therefore, condition (2) is proved.

On the other hand, to prove condition (4), assume that $p_{\bar{t}}^m \approx p_{\bar{t}_1}^+$ for $0 \leq m < \mu(\bar{t})$. Then it follows from assertion (2) of Lemma 6.1 that $\bar{\Gamma}_\tau^{(2)} \cap P(\tau)$ contains either $\alpha_{\bar{t}, \bar{t}'}^k$ with $\theta_{i, i'}(k) + \mu(\bar{t}') = m$ and with $\theta_{i, i'}(k) \geq 0$, or $\alpha_{\bar{t}', \bar{t}}^k$ with $\theta_{i', i}(k) + m = \mu(\bar{t}')$ and with $\theta_{i', i}(k) > 0$. In the latter case, we have $\theta_{i', i}(k) + \mu(\bar{t}) > \theta_{i', i}(k) + m = \mu(\bar{t}')$, and thus $\alpha_{\bar{t}', \bar{t}}^k \in \Gamma_\tau^{(2)}$, which is a contradiction. In the former case, one has $\theta_{i, i'}(k) + \mu(\bar{t}') = m < \mu(\bar{t})$. If $\theta_{i, i'}(k) > 0$, \bar{t}' and \bar{t} satisfy $\bar{t}' \prec \bar{t}$. Similarly, if $\theta_{i, i'}(k) = 0$, they also satisfy $\bar{t}' \prec \bar{t}$, since the inequality $\mu(\bar{t}') = m < \mu(\bar{t})$ and the assumption $\alpha_{\bar{t}, \bar{t}'}^k \notin \Gamma_\tau^{(2)}$ together yield $\bar{t}' \prec_i \bar{t}$. Therefore, condition (4) is proved, and the total order \prec satisfies conditions (1)–(4) in Proposition 6.2.

Conversely, assume that there is a total order \prec on $\mathcal{K}(n)$ satisfying conditions (1)–(4), or \bar{f} is a realization of τ . We claim that there is an n -tuple $(\prec_i) \in \mathcal{T}(\tau)$ such that if $p_{\bar{t}'}^- < p_{\bar{t}}^-$, and thus $\bar{t}' \not\geq \bar{t}$, then $\bar{t}' \prec_i \bar{t}$. In order to prove the claim, it is enough to show that for any $p_{\bar{t}'}^- < p_{\bar{t}}^-$, either $\mu(\bar{t}') = \mu(\bar{t})$ and $\bar{t}'_1 \prec \bar{t}_1$, or $\mu(\bar{t}') < \mu(\bar{t})$ holds. First, we assume the contrary that $\mu(\bar{t}') > \mu(\bar{t})$. As $p_{\bar{t}'}^- \approx p_{\bar{t}}^-$, we have $p_{\bar{t}'}^{\mu(\bar{t})} \approx p_{\bar{t}_1}^+$ and hence $\bar{t} \not\geq \bar{t}'$ from condition (4), which is a contradiction. On the other hand, if $\mu(\bar{t}') = \mu(\bar{t})$, then it follows that $p_{\bar{t}_1}^+ \approx p_{\bar{t}'_1}^+$. From the relation $\bar{t}' \not\geq \bar{t}$ and condition (2), one has $p_{\bar{t}'_1}^+ < p_{\bar{t}_1}^+$ and $p_{\bar{t}'_1}^- < p_{\bar{t}_1}^-$, which yields $\bar{t}'_1 \prec \bar{t}_1$. Therefore, we establish the claim and, in what follows, fix $(\prec_i) \in \mathcal{T}(\tau)$ mentioned in the claim.

Next, we assume the contrary that there is a periodic root $\alpha_{\bar{t}, \bar{t}'}^k \in \bar{\Gamma}_\tau^{(2)} \cap P(\tau)$, which means that $p_{\bar{t}}^m \approx p_{\bar{t}'}^-$ with $m = \theta_{i, i'}(k) \leq \mu(\bar{t})$. Note that if $\theta_{i, i'}(k) > 0$ then the relation $\bar{t}' \not\geq \bar{t}$ holds from condition (3), but if $\mu(\bar{t}) < \mu(\bar{t}') + \theta_{i, i'}(k)$ then the relation $\bar{t} \not\geq \bar{t}'$ holds, since $p_{\bar{t}'}^{m'} \approx p_{\bar{t}_1}^+$ for $m' + \theta_{i, i'}(k) = \mu(\bar{t})$ and thus for $m' < \mu(\bar{t}')$. Now, we assume that $\theta_{i, i'}(k) > 0$ and so $\bar{t}' \not\geq \bar{t}$. Since $\alpha_{\bar{t}, \bar{t}'}^k \notin \check{\Gamma}_\tau^{(2)}$ and $\mu(\bar{t}) \geq \mu(\bar{t}') + \theta_{i, i'}(k)$, one has $\mu(\bar{t}) = \mu(\bar{t}') + \theta_{i, i'}(k)$ and $\bar{t}_1 \prec_i \bar{t}'_1$. This

means that the relations $p_{\bar{t}_1}^- < p_{\bar{t}'_1}^-$ and $p_{\bar{t}_1}^+ < p_{\bar{t}'_1}^+$ hold. Therefore, one has $\bar{t} \not\preceq \bar{t}'$ from condition (2), which is a contradiction. On the other hand, assume that $\theta_{i,i'}(k) = 0$ and $\mu(\bar{t}) < \mu(\bar{t}')$, which leads to $\bar{t}' \prec_i \bar{t}$ as $\alpha_{\bar{t},\bar{t}'}^k \notin \check{\Gamma}_\tau^{(2)}$. By the first equality, one has $p_{\bar{t}}^- \approx p_{\bar{t}'}^-$, and by the second inequality and condition (3), one has $p_{\bar{t}}^{\mu(\bar{t})} \approx p_{\bar{t}_1}^+$ and $\bar{t} \not\preceq \bar{t}'$, which is a contradiction. Finally, assume that $\theta_{i,i'}(k) = 0$, $\mu(\bar{t}) = \mu(\bar{t}')$ and $\bar{t}' \prec_i \bar{t}$. In this case, we have $p_{\bar{t}}^- \approx p_{\bar{t}'}^-$ and $p_{\bar{t}_1}^+ \approx p_{\bar{t}'_1}^+$. Since $\alpha_{\bar{t},\bar{t}'}^k \notin \check{\Gamma}_\tau^{(2)}$, one has $\bar{t}_1 \prec_{i_1} \bar{t}'_1$, $p_{\bar{t}_1}^- < p_{\bar{t}'_1}^-$ and $p_{\bar{t}_1}^+ < p_{\bar{t}'_1}^+$. This shows that $\bar{t} \not\preceq \bar{t}'$ from condition (2), which is a contradiction. Summing up this discussion, τ satisfies condition (36) and the proposition is established. \square

Proposition 6.4 *Let τ be an orbit data satisfying $\lambda(w_\tau) > 1$ and condition (31), and \bar{f} be the tentative realization mentioned in Proposition 5.7. Then, there is an orbit data $\check{\tau}$ such that $\delta = \lambda(w_\tau) = \lambda(w_{\check{\tau}})$ and \bar{f} is a realization of $\check{\tau}$. In particular, $\check{\tau}$ satisfies condition (36).*

Proof. Under the notation of Lemma 4.7, assume that there is a sequence $\bar{t}^{(1)} \prec \bar{t}^{(2)} \prec \dots \prec \bar{t}^{(j)} \in \mathcal{K}(n)$ such that $(p_{\bar{t}^{(\ell)}}^-, p_{\sigma(\bar{t}^{(\ell)})}^+)$ is a proper pair of $\bar{g}_{\ell-1}$ with length $\mu(\bar{t}^{(\ell)})$ for any $\ell = 1, \dots, j$, and $(p_{\bar{t}}^-, p_{\sigma(\bar{t})}^+)$ are not proper pairs of \bar{g}_j with length $\mu(\bar{t})$ for any $\bar{t} \in \mathcal{K}(\bar{g}_j) = \mathcal{K}(n) \setminus \{\bar{t}^{(1)} \dots \bar{t}^{(j)}\}$. If $j = 3n$, then \bar{f} is a realization of τ and the proposition is already proved. Otherwise, there is a pair $(\bar{t}', \bar{t}'') \in \mathcal{K}(\bar{g}_j) \times \mathcal{K}(\bar{g}_j)$ such that

- (1) $p_{\bar{t}''}^{m'} \approx p_{\bar{t}'}^-$ with $(\bar{t}', \bar{t}'', m') \neq (\bar{t}', \bar{t}', 0)$ and $\mu(\bar{t}'') - m' = \min \{ \mu(\bar{t}) - m \mid p_{\bar{t}}^m \approx p_{\bar{t}''}^m, \bar{t}, \bar{t}'' \in \mathcal{K}(\bar{g}_j), (\bar{t}, \bar{t}'', m) \neq (\bar{t}, \bar{t}, 0) \},$
- (2) if $p_{\bar{t}''}^m \approx p_{\bar{t}'}^-$ satisfy $\mu(\bar{t}'') - m' = \mu(\bar{t}') - m$ and thus $i' = i$, then $p_{\bar{t}'}^- < p_{\bar{t}}^-$,
- (3) if $p_{\bar{t}''}^m \approx p_{\bar{t}'}^-$ satisfy $\mu(\bar{t}'') - m' = \mu(\bar{t}) - m$ and thus $i'' = i_1$, then $p_{\bar{t}'_1}^+ < p_{\bar{t}_1}^+$.

Let $(v, s_\tau) \in (\mathbb{C}^{3n} \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ be the unique solution of (13) and (14) as in Corollary 3.4, and denote by $u_{\bar{t}}$ the value given in (35) with $d = \delta$ and $s_j = (s_\tau)_j$.

If $\bar{t}' \neq \bar{t}''$, then put $\check{\tau} = (n, \check{\sigma}, \check{\mu})$, where $\check{\sigma} : \mathcal{K}(n) \rightarrow \mathcal{K}(n)$ and $\check{\mu} : \mathcal{K}(n) \rightarrow \mathbb{Z}_{\geq 0}$ are given by

$$\check{\sigma}(\bar{t}) := \begin{cases} \sigma(\bar{t}'') & (\bar{t} = \bar{t}') \\ \sigma(\bar{t}') & (\bar{t} = \bar{t}'') \\ \sigma(\bar{t}) & (\text{otherwise}), \end{cases} \quad \check{\mu}(\bar{t}) := \begin{cases} \mu(\bar{t}'') - m' & (\bar{t} = \bar{t}') \\ \mu(\bar{t}') + m' & (\bar{t} = \bar{t}'') \\ \mu(\bar{t}) & (\text{otherwise}). \end{cases}$$

Since $p_{\bar{t}''}^{m'} \approx p_{\bar{t}'}^-$, one has $p_{\bar{t}''}^{\check{\mu}(\bar{t}'')} = p_{\bar{t}''}^{\mu(\bar{t}') + m'} \approx p_{\bar{t}'}^{\mu(\bar{t}')} \approx p_{\sigma(\bar{t}')}^+ = p_{\sigma(\bar{t}'')}^+$ and $p_{\bar{t}'}^{\check{\mu}(\bar{t}')} = p_{\bar{t}'}^{\mu(\bar{t}'') - m'} \approx p_{\bar{t}''}^{\mu(\bar{t}'')} \approx p_{\sigma(\bar{t}'')}^+ = p_{\sigma(\bar{t}')}^+$, which yield $\delta^{\check{\kappa}(\bar{t}')-1} \cdot v_{\bar{t}'} = u_{\sigma(\bar{t}')} and $\delta^{\check{\kappa}(\bar{t}'')-1} \cdot v_{\bar{t}''} = u_{\sigma(\bar{t}'')}$. For $\bar{t} \neq \bar{t}', \bar{t}''$, the equation $\delta^{\check{\kappa}(\bar{t})-1} \cdot v_{\bar{t}} = u_{\sigma(\bar{t})}$ leads to $\delta^{\check{\kappa}(\bar{t})-1} \cdot v_{\bar{t}} = u_{\sigma(\bar{t})}$. This means that $v_{\bar{t}} = \delta^{\check{\kappa}(\bar{t})} \cdot v_{\bar{t}} + (\delta - 1) \cdot (s_\tau)_{i_{\check{\sigma}(\bar{t})}}$ for any $\bar{t} \in \mathcal{K}(n)$, where $\bar{t}_k^{\check{\sigma}} = (i_k^{\check{\sigma}}, \iota_k^{\check{\sigma}}) := \check{\sigma}^k(\bar{t})$. As (δ, v, s_τ) satisfies (13), we have $\mathcal{A}_{\check{\tau}}(\delta) s_\tau = 0$, and thus $\delta = \lambda(w_\tau) = \lambda(w_{\check{\tau}})$. Moreover, $(p_{\bar{t}'}^-, p_{\sigma(\bar{t}')}^+)$ is a proper pair of \bar{g}_j with length $\check{\mu}(\bar{t}')$. Indeed, from conditions (2) and (3), $p_{\bar{t}'}^-$ and $p_{\sigma(\bar{t}')}^+$ are proper points, and from the minimality of the number $\mu(\bar{t}'') - m'$, $p_{\bar{t}''}^m$ satisfies $p_{\bar{t}''}^m \not\approx p_{\bar{t}'}^+$ for any $0 \leq m < \mu(\bar{t}')$ and $\bar{t} \in \mathcal{K}(\bar{g}_j)$, and also satisfies $p_{\bar{t}''}^m \not\approx p_{\bar{t}'}^-$ for any $0 < m \leq \mu(\bar{t}')$ and $\bar{t} \in \mathcal{K}(\bar{g}_j)$. Furthermore, $(p_{\bar{t}^{(\ell)}}^-, p_{\sigma(\bar{t}^{(\ell)})}^+)$ remains a proper pair of $\bar{g}_{\ell-1}$ with length $\check{\mu}(\bar{t}^{(\ell)})$ for $\ell = 1, \dots, j$, since $\check{\sigma}(\bar{t}^{(\ell)}) = \sigma(\bar{t}^{(\ell)})$, $\check{\mu}(\bar{t}^{(\ell)}) = \mu(\bar{t}^{(\ell)})$, and the indeterminacy points of $\bar{g}_{\ell-1}$ are invariant under the change of orbit data.$

On the other hand, if $\bar{\tau}'' = \bar{\tau}'$ and $m' > 0$, then the new orbit data $\tilde{\tau} = (n, \sigma, \tilde{\mu})$ is defined by $\tilde{\mu}(\bar{\tau}') := \mu(\bar{\tau}') - m'$, and $\tilde{\mu}(\bar{\tau}) := \mu(\bar{\tau})$ if $\bar{\tau} \neq \bar{\tau}'$. Note that for $m' > 0$, the relation $p_{\bar{\tau}'}^{m'} = p_{\bar{\tau}'}^-$ yields $\delta^{k'} \cdot v_{\bar{\tau}'} = v_{\bar{\tau}'}$ for some $k' > 0$. Since δ is not a root of unity, we have $v_{\bar{\tau}'} = 0$. Therefore, v satisfies (13) and (14) with $d = \delta$ and with the orbit data $\tilde{\tau}$. This means that $\delta = \lambda(w_\tau) = \lambda(w_{\tilde{\tau}})$. Similarly, $(p_{\bar{\tau}'}^-, p_{\sigma(\bar{\tau}')}^+)$ is a proper pair of \bar{g}_j with length $\tilde{\mu}(\bar{\tau}')$, and $(p_{\bar{\tau}^{(\ell)}}^-, p_{\sigma(\bar{\tau}^{(\ell)})}^+)$ remains a proper pair of $\bar{g}_{\ell-1}$ with length $\tilde{\mu}(\bar{\tau}^{(\ell)})$ for $\ell = 1, \dots, j$.

Thus, either \bar{f} is a realization of $\tilde{\tau}$, or we can repeat the above argument to construct a realization. When $\tilde{\tau}$ admits the realization \bar{f} , it follows from Proposition 6.3 that $\tilde{\tau}$ satisfies condition (36), and so the proposition is established. \square

Proposition 6.5 *Let τ be an orbit data satisfying conditions (1) and (2) in Theorem 1.5. Then we have $\bar{\Gamma}_\tau^{(2)} \cap P(\tau) = \{\alpha_{\bar{\tau}, \bar{\tau}'}^0 \mid i_m = i'_m, \kappa(\bar{\tau}_m) = \kappa(\bar{\tau}'_m), m \geq 0\}$. In addition, if τ satisfies condition (3) in Theorem 1.5, then it also satisfies condition (36).*

The proof of this proposition is given in Section 7. We are now in a position to establish the main theorems.

Proofs of Theorems 1.3–1.5. Theorem 1.3 is an immediate consequence of Propositions 4.9, 5.7 and 6.3. We notice that the points $\{p_{\bar{\tau}}^m \mid \bar{\tau} \in \mathcal{K}(n), m = \theta_{i,0}(k), 1 \leq k \leq \kappa(\bar{\tau})\}$, which are blown up by π_τ , lie on C^* . Moreover, Theorem 1.4 follows from Propositions 5.8 and 6.4, and Theorem 1.5 follows from Propositions 5.9 and 6.5. \square

Proof of Corollary 1.6. For any value $\lambda \neq 1 \in \Lambda$, Theorem 1.4 and Proposition 2.6 show that there is an orbit data τ such that $\lambda = \lambda(w_\tau)$ and τ satisfies the realizability condition (5). In particular, the automorphism F_τ mentioned in Theorem 1.3 has entropy $h_{\text{top}}(F_\tau) = \log \lambda > 0$. Note that when $\lambda = 1 \in \Lambda$, the automorphism $\text{id}_{\mathbb{P}^2} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ satisfies $\lambda(\text{id}_{\mathbb{P}^2}^*) = \lambda = 1$ and $h_{\text{top}}(\text{id}_{\mathbb{P}^2}) = 0$. On the other hand, from Proposition 4.3, the entropy of any automorphism $F : X \rightarrow X$ is given by $h_{\text{top}}(F) = \log \lambda$ for some $\lambda \in \Lambda$. Therefore, Corollary 1.6 is proved. \square

7 Proof of Realizability with Estimates

As is seen in Section 6, Propositions 5.9 and 6.5 prove Theorem 1.5, or the realizability of orbit data. In this section, we prove these propositions by applying some estimates mentioned below. To this end, we fix an orbit data τ satisfying conditions (1) and (2) in Theorem 1.5.

Lemma 7.1 *If $d > 1$, then for any $j \in \{1, \dots, n\}$ and $\bar{\tau} \in \mathcal{K}(n)$, we have*

$$-\frac{1}{d^2 + d + 1} \leq \bar{c}_{\bar{\tau}, j}(d) \leq 0,$$

where $\bar{c}_{\bar{\tau}, j}(d)$ is given by (18).

Proof. In view of equation (17), $\bar{c}_{\bar{\tau}, j}(d)$ may be expressed as either $\bar{c}_{\bar{\tau}, j}(d) = 0$, or $\bar{c}_{\bar{\tau}, j}(d) = -(d-1) \cdot d^{\eta_1} / (d^\eta - 1)$ with $\eta_1 + 3 \leq \eta$, or $\bar{c}_{\bar{\tau}, j}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2}) / (d^\eta - 1)$ with $\eta_1 + 3 \leq \eta_2$ and $\eta_2 + 3 \leq \eta$, or $\bar{c}_{\bar{\tau}, j}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2} + d^{\eta_3}) / (d^\eta - 1)$ with $\eta_1 + 3 \leq \eta_2$, $\eta_2 + 3 \leq \eta_3$

and $\eta_3 + 3 \leq \eta$, since $\#\{m \mid i_m = j\} \leq \#\{(j, 1), (j, 2), (j, 3)\} = 3$. We only consider the case $\bar{c}_{\ell,j}(d) = -(d-1) \cdot (d^{\eta_1} + d^{\eta_2}) / (d^{\eta} - 1)$ as the remaining cases can be treated in the same manner. Since $d > 1$, the inequality $\bar{c}_{\ell,j}(d) < 0$ is trivial. Moreover, one has

$$\frac{\bar{c}_{\ell,j}(d)}{d-1} = -\frac{d^{\eta_1} + d^{\eta_2}}{d^{\eta} - 1} \geq -\frac{d^{\eta_2-3} + d^{\eta_2}}{d^{\eta} - 1} \geq -\frac{d^{\eta-6} + d^{\eta-3}}{d^{\eta} - 1} = (1 - \frac{1}{d^{\eta} - 1})(d^{-6} + d^{-3}) \geq -\frac{1}{d^3 - 1}.$$

Thus the lemma is established. \square

Since $c_{i,j}(d) = -\sum_{\iota=1}^3 \bar{c}_{(i,\iota),j}(d)$ from (19), the above lemma leads to the inequality

$$0 \leq c_{i,j}(d) \leq \gamma_d, \quad \gamma_d := \frac{3}{1 + d + d^2}.$$

Note that for any $d \geq 2$ and any $0 \leq x_{i,j} \leq \gamma_d$, each diagonal entry $\mathcal{A}_n(d, x)_{i,i}$ of $\mathcal{A}_n(d, x)$ is positive and each non-diagonal entry $\mathcal{A}_n(d, x)_{i,j}$ with $i \neq j$ is negative, where $\mathcal{A}_n(d, x)$ is the matrix given in (20). Let $\bar{\mathcal{A}}_n(d, x)_{i,j}$ be the (i, j) -cofactor of the matrix $\mathcal{A}_n(d, x)$. Then, the relation $|\mathcal{A}_n(d, x)| = \sum_{i=1}^n \bar{\mathcal{A}}_n(d, x)_{i,i} \cdot \mathcal{A}_n(d, x)_{i,i}$ holds for any $j = 1, \dots, n$, where $|\mathcal{A}_n(d, x)|$ is the determinant of the matrix $\mathcal{A}_n(d, x)$.

Lemma 7.2 *For any $n \geq 2$, the following inequalities hold:*

$$\begin{cases} \bar{\mathcal{A}}_n(d, x)_{i,j} > 0, & (d > 2^n - 1, 0 \leq x_{i,j} \leq \gamma_d) \\ |\mathcal{A}_n(d, x)| > 0, & (d > 2^n, 0 \leq x_{i,j} \leq \gamma_d) \\ |\mathcal{A}_n(2^n - 1, x)| < 0, & (0 \leq x_{i,j} \leq \gamma_d). \end{cases}$$

Proof. We prove the inequalities by induction on n . For $n = 2$, the first inequality holds since

$$\bar{\mathcal{A}}_2(d, x)_{i,j} = \begin{cases} -\mathcal{A}_2(d, x)_{j,i} > 0 & (i \neq j) \\ \mathcal{A}_2(d, x)_{i+1,i+1} > 0 & (i = j \in \mathbb{Z}/2\mathbb{Z}). \end{cases}$$

As $\gamma_d < \frac{3}{13}$ when $d > 3$, the remaining inequalities follow from the estimates

$$\begin{cases} |\mathcal{A}_2(3, x)| = (1 + x_{1,1})(1 + x_{2,2}) - (1 - x_{2,1})(3 - x_{1,2}) < (1 + \frac{3}{13})^2 - (1 - \frac{3}{13})(3 - \frac{3}{13}) < 0 \\ |\mathcal{A}_2(d, x)| = (d - 2 + x_{1,1})(d - 2 + x_{2,2}) - (1 - x_{2,1})(d - x_{1,2}) > 2^2 - 1 \cdot 4 = 0. \end{cases}$$

Therefore, the lemma is proved when $n = 2$. Assume that the inequalities hold when $n = l - 1$. A little calculation shows that $\bar{\mathcal{A}}_{i,j} := \bar{\mathcal{A}}_l(d, x)_{i,j}$ can be expressed as

$$\bar{\mathcal{A}}_{i,j} = \begin{cases} -\left\{ \sum_{k=1}^{i-1} \bar{\mathcal{A}}_{l-1}(d, x^i)_{k,j-1} \cdot \mathcal{A}_l(d, x)_{k,i} + \sum_{k=i+1}^l \bar{\mathcal{A}}_{l-1}(d, x^i)_{k-1,j-1} \cdot \mathcal{A}_l(d, x)_{k,i} \right\} & (i < j) \\ -\left\{ \sum_{k=1}^{i-1} \bar{\mathcal{A}}_{l-1}(d, x^i)_{k,j} \cdot \mathcal{A}_l(d, x)_{k,i} + \sum_{k=i+1}^l \bar{\mathcal{A}}_{l-1}(d, x^i)_{k-1,j} \cdot \mathcal{A}_l(d, x)_{k,i} \right\} & (i < j) \\ |\mathcal{A}_{l-1}(d, x^i)| & (i = j), \end{cases}$$

where x^i is the $(l-1, l-1)$ -matrix obtained from x by removing the i -th row and column vectors. Therefore, the first assertion follows from the induction hypothesis. Moreover, since $|\mathcal{A}_l(d, x)| = \sum_{i=1}^l \bar{\mathcal{A}}_l(d, x)_{i,i} \cdot \mathcal{A}_l(d, x)_{i,i}$, the bounds

$$|\mathcal{A}_l(d, (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_l))| \leq |\mathcal{A}_l(d, x)| \leq |\mathcal{A}_l(d, (x_1, \dots, x_{j-1}, \bar{\gamma}_d, x_{j+1}, \dots, x_l))|$$

hold for any j , where $\bar{\gamma}_d$ is the column vector having each component equal to γ_d . Thus, we have

$$|\mathcal{A}_l(d, x)| \geq |\mathcal{A}_l(d, (0, \dots, 0))| = (d-1)^{l-1}(d-2^l) > 0, \quad (d > 2^l),$$

$$|\mathcal{A}_l(2^l - 1, x)| \leq |\mathcal{A}_l(2^l - 1, (\bar{\gamma}_{2^l-1}, \dots, \bar{\gamma}_{2^l-1}))| = -\frac{(2^l - 2)^{l+1}}{2^{2l} - 2^l + 1} < 0,$$

which show that the assertions are verified when $n = l$. Therefore, the induction is complete, and the lemma is established. \square

Let s_τ be the unique solution of equation (21) with $d = \delta$.

Lemma 7.3 *For any $1 \leq i \leq n-1$, the ratio $(s_\tau)_{i+1}/(s_\tau)_i$ satisfies*

$$z_1(n) < \frac{(s_\tau)_{i+1}}{(s_\tau)_i} < z_2(n),$$

where

$$z_1(n) := \frac{2^{n-1}(2^n + 2)}{2^{2n} + 2^{n+1} + 6}, \quad z_2(n) := \frac{2^{2n-1} + 2^n + 3}{2^{2n} + 2^{n+1} + 3}.$$

Proof. For each $k_1, k_2 \geq 0$ with $k_1 + k_2 \leq n-2$, let $\mathcal{A}_n^{k_1, k_2}(\delta)$ be the $n \times n$ matrix defined inductively as follows. First, put $\mathcal{A}_n^{0,0}(\delta) := \mathcal{A}_\tau(\delta)$. Next, let $\mathcal{A}_n^{k_1, 0}(\delta)$ be the matrix obtained from $\mathcal{A}_n^{k_1-1, 0}(\delta)$ by replacing the i -th row of $\mathcal{A}_n^{k_1-1, 0}(\delta)$ with the sum of the i -th row and the k_1 -th row multiplied by $-\mathcal{A}_n^{k_1-1, 0}(\delta)_{i, k_1}/\mathcal{A}_n^{k_1-1, 0}(\delta)_{k_1, k_1}$, where i runs from $k_1 + 1$ to n . Finally, let $\mathcal{A}_n^{k_1, k_2}(\delta)$ be the matrix obtained from $\mathcal{A}_n^{k_1, k_2-1}(\delta)$ by replacing the i -th row of $\mathcal{A}_n^{k_1, k_2-1}(\delta)$ with the sum of the i -th row and the $(n - k_2 + 1)$ -th row multiplied by $-\mathcal{A}_n^{k_1, k_2-1}(\delta)_{i, n-k_2+1}/\mathcal{A}_n^{k_1, k_2-1}(\delta)_{n-k_2+1, n-k_2+1}$, where i runs from $k_1 + 1$ to $n - k_2$. Therefore, each entry of $\mathcal{A}_n^{k_1, k_2}(\delta)$ may be expressed as

$$\mathcal{A}_n^{k_1, k_2}(\delta)_{i,j} = \begin{cases} \delta_{i,j} + \xi_{i,j}^{i-1} & (i \leq k_1 \text{ and } i \leq j) \\ \delta_{i,j} + \xi_{i,j}^{k_1+k_2} & (k_1 + 1 \leq i, j \leq n - k_2) \\ \delta_{i,j} + \xi_{i,j}^{k_1+n-i} & (n - k_2 + 1 \leq i \text{ and } k_1 + 1 \leq j \leq i) \\ 0 & (\text{otherwise}), \end{cases}$$

where

$$\delta_{i,j} = \begin{cases} \delta - 2 & (i = j) \\ -1 & (i > j) \\ -\delta & (i < j), \end{cases}$$

and $\xi_{i,j}^k$ is given inductively by

$$\xi_{i,j}^0 = c_{i,j}(\delta), \quad \xi_{i,j}^{k+1} = \begin{cases} \xi_{i,j}^k - \frac{(1 - \xi_{i,k}^k)(\delta - \xi_{k,j}^k)}{\delta - 2 + \xi_{k,k}^k} & (k < k_1) \\ \xi_{i,j}^k - \frac{(\delta - \xi_{i, n-k+k_1}^k)(1 - \xi_{n-k+k_1, j}^k)}{\delta - 2 + \xi_{n-k+k_1, n-k+k_1}^k} & (k \geq k_1). \end{cases}$$

Moreover, it is seen that $\xi_{i,j}^k$ satisfies the estimates

$$-\frac{(2^k - 1)\delta}{\delta - 2^k} \leq \xi_{i,j}^k \leq -\bar{\xi}_k, \quad \bar{\xi}_k := \frac{(2^k - 1)\delta - (2^k\delta - 1)\gamma_\delta}{(\delta - 2^k) + (2^k - 1)\gamma_\delta}.$$

Note that s_τ satisfies $\mathcal{A}_n^{k_1, k_2}(\delta) s_\tau = 0$ for any $k_1, k_2 \geq 0$. In particular, one has $\mathcal{A}_n^{i-1, n-i-1}(\delta) s_\tau = 0$, the i -th and $(i+1)$ -th components of which are given by

$$\begin{cases} (\delta - 2 + \xi_{i,i}^{n-2})(s_\tau)_i + (-\delta + \xi_{i,i+1}^{n-2})(s_\tau)_{i+1} = 0, \\ (-1 + \xi_{i+1,i}^{n-2})(s_\tau)_i + (\delta - 2 + \xi_{i+1,i+1}^{n-2})(s_\tau)_{i+1} = 0. \end{cases}$$

Therefore, we have

$$\frac{(s_\tau)_{i+1}}{(s_\tau)_i} = \frac{\delta - 2 + \xi_{i,i}^{n-2}}{\delta - \xi_{i,i+1}^{n-2}} < \frac{\delta - 2 - \bar{\xi}_{n-2}}{\delta + \bar{\xi}_{n-2}} = \frac{2\delta^2 - (2^n - 4)\delta - (2^{n+1} - 6)}{2(\delta^2 + 2\delta + 3)},$$

the righthand side of which is monotone increasing with respect to δ , and thus is less than $z_2(n)$ since $\delta < 2^n$. In a similar manner, we have

$$\frac{(s_\tau)_{i+1}}{(s_\tau)_i} = \frac{1 - \xi_{i+1,i}^{n-2}}{\delta - 2 + \xi_{i+1,i+1}^{n-2}} > \frac{1 + \bar{\xi}_{n-2}}{\delta - 2 - \bar{\xi}_{n-2}} = \frac{2^{n-2}(1 - \gamma_\delta)}{\delta - 2^{n-1} + (2^{n-1} - 1)\gamma_\delta} > z_1(n).$$

Thus, the lemma is established. \square

We remark that the functions $z_1(n)$ and $z_2(n)$ satisfy

$$0 < z_1(n) < \frac{1}{2} < z_2(n) < 1.$$

Proof of Proposition 5.9. Recall that $\chi_\tau(d) = |\mathcal{A}_\tau(d)|$. From Lemma 7.2, one has $\chi_\tau(2^n - 1) < 0$ and $\chi_\tau(2^n - 1) > 0$. Therefore, there is a real number δ such that $\chi_\tau(\delta) = 0$ and $2^n - 1 < \delta < 2^n$. Moreover, it follows from Lemma 7.3 that $(s_\tau)_j \neq 0$ for any j . Thus Lemma 5.6 yields $\Gamma_\tau^{(1)} \cap P(\tau) = \emptyset$. \square

Next we prove Proposition 6.5.

Lemma 7.4 *For any $n \geq 2$, we have the following two inequalities:*

- (1) $g_1(n) < 0$, where $g_1(n) := \frac{1}{\delta^3 - 1} + 1 - \delta \cdot z_1(n)^{n-1}$,
- (2) $g_2(n) > 0$, where $g_2(n) := z_1(n)^{n-2} - z_2(n)^{n-1} - \frac{1}{\delta^3 - 1}$.

Proof. First, we claim that the following inequality holds:

$$z_1(n)^{n-1} > \frac{1}{2^{n-1}} - (n-1) \left(\frac{1}{2^{3n-4}} + \frac{1}{2^{4n-3}} \right). \quad (38)$$

Indeed, since

$$\left(1 - \frac{1}{2^{n-1}} - \frac{1}{2^{2n-1}}\right) \left(1 + \frac{1}{2^{n-1}} + \frac{6}{2^{2n}}\right) = 1 - \frac{1}{2^{4n-2}}(2^{n+2} + 3) \leq 1,$$

one has

$$z_1(n) \geq \frac{1}{2} \left(1 + \frac{1}{2^{n-1}}\right) \left(1 - \frac{1}{2^{n-1}} - \frac{1}{2^{2n-1}}\right) = \frac{1}{2} \left\{1 - \left(\frac{3}{2^{2n-1}} + \frac{1}{2^{3n-2}}\right)\right\} > \frac{1}{2} \left\{1 - \left(\frac{1}{2^{2n-3}} + \frac{1}{2^{3n-2}}\right)\right\}.$$

Therefore, the claim holds from the Bernoulli inequality, namely, $(1+x)^n \geq 1+nx$ for any $x \geq -1$. By using inequality (38), we prove the two inequalities in the lemma.

In order to prove assertion (1), we consider the function of n :

$$\check{g}_1(n) := \frac{1}{(2^n - 1)^3 - 1} + 1 - (2^n - 1) \cdot z_1(n)^{n-1}.$$

Then the inequality $g_1(n) < \check{g}_1(n)$ holds since $\delta > 2^n - 1$. Moreover, as $\check{g}_1(2) < 0$, one has $g_1(2) < 0$. On the other hand, when $n \geq 3$, inequality (38) yields

$$\begin{aligned} \check{g}_1(n) &< \frac{(2^n - 1)^3}{(2^n - 1)^3 - 1} - \frac{2^n - 1}{2^{n-1}} \left\{ 1 - (n-1) \left(\frac{1}{2^{2n-3}} + \frac{1}{2^{3n-2}} \right) \right\} \\ &< \frac{2^n - 1}{2^{n-1}} \left(-1 + \frac{2^{n-1}(2^n - 1)^2}{(2^n - 1)^3 - 1} + \frac{n-1}{2^{2n-3}} + \frac{n-1}{2^{3n-2}} \right). \end{aligned}$$

Since the terms $\frac{2^{n-1}(2^n - 1)^2}{(2^n - 1)^3 - 1}$, $\frac{n-1}{2^{2n-3}}$ and $\frac{n-1}{2^{3n-2}}$ are monotone decreasing with respect to n , the function $-1 + \frac{2^{n-1}(2^n - 1)^2}{(2^n - 1)^3 - 1} + \frac{n-1}{2^{2n-3}} + \frac{n-1}{2^{3n-2}}$ is maximized when $n = 3$, which is negative. Therefore, we have $\check{g}_1(n) < 0$, and thus $g_1(n) < 0$.

Finally, in order to prove assertion (2), we consider the function of n :

$$\check{g}_2(n) := z_1(n)^{n-2} - z_2(n)^{n-1} - \frac{1}{(2^n - 1)^3 - 1}.$$

Then the inequality $g_2(n) > \check{g}_2(n)$ holds since $\delta > 2^n - 1$. Moreover, as $\check{g}_2(2), \check{g}_2(3) > 0$, one has $g_2(2), g_2(3) > 0$. On the other hand, when $n \geq 4$, $\check{g}_2(n)$ can be estimated as

$$\begin{aligned} \check{g}_2(n) &= z_1(n)^{n-2} (1 - z_1(n)) - (z_2(n)^{n-1} - z_1(n)^{n-1}) - \frac{1}{(2^n - 1)^3 - 1} \\ &\geq z_1(n)^{n-2} (1 - z_1(n)) - (n-1)(z_2(n) - z_1(n))z_2(n)^{n-2} - \frac{1}{(2^n - 1)^3 - 1}, \end{aligned}$$

where the last inequality follows from the general inequality $x^n - y^n \leq n(x - y)x^{n-1}$ for any $x \geq y \geq 0$. Since $z_2(n) - z_1(n) = \frac{9}{2} \frac{2^{2n} + 2^{n+1} + 4}{(2^{2n} + 2^{n+1} + 3)(2^{2n} + 2^{n+1} + 6)} < \frac{9}{2} \frac{1}{2^{2n} + 2^{n+1} + 3} < \frac{9}{8} \frac{1}{2^{2(n-1)}}$, and $z_2(n) = \frac{1}{2} + \frac{3}{2} \frac{1}{2^{2n} + 2^{n+1} + 3}$ is monotone decreasing with respect to n , and thus is less than $\frac{13}{24}$, we have

$$(n-1)(z_2(n) - z_1(n))z_2(n)^{n-2} < (n-1) \frac{9}{8} \left(\frac{13}{24} \right)^{n-2} \frac{1}{2^{2(n-1)}} < \frac{1}{2^{2(n-1)}},$$

where we use the fact that the function $(n-1) \frac{9}{8} \left(\frac{13}{24} \right)^{n-2}$ is monotone decreasing and is less than 1. Moreover, as $1 - z_1(n) > z_1(n)$, one has

$$\begin{aligned} \check{g}_2(n) &> z_1(n)^{n-1} - \frac{1}{2^{2(n-1)}} - \frac{1}{(2^n - 1)^3 - 1} \\ &> \frac{1}{2^{n-1}} \left\{ 1 - (n-1) \left(\frac{1}{2^{2n-3}} + \frac{1}{2^{3n-2}} \right) \right\} - \frac{1}{2^{2(n-1)}} - \frac{1}{(2^n - 1)^3 - 1} \\ &= \frac{1}{2^{n-1}} \left(1 - \frac{n-1}{2^{2n-3}} - \frac{n-1}{2^{3n-2}} - \frac{1}{2^{n-1}} - \frac{2^{n-1}}{(2^n - 1)^3 - 1} \right). \end{aligned}$$

Since the terms $\frac{n-1}{2^{2n-3}}$, $\frac{n-1}{2^{3n-2}}$, $\frac{1}{2^{n-1}}$ and $\frac{2^{n-1}}{(2^n - 1)^3 - 1}$ are monotone decreasing with respect to n , the function $1 - \frac{n-1}{2^{2n-3}} - \frac{n-1}{2^{3n-2}} - \frac{1}{2^{n-1}} - \frac{2^{n-1}}{(2^n - 1)^3 - 1}$ is minimized when $n = 4$, which is positive. Therefore, we have $\check{g}_2(n) > 0$ and thus $g_2(n) > 0$, and so the proof is complete. \square

Lemma 7.5 Assume that $v_{\bar{\tau}}(\delta) = \delta^k \cdot v_{\bar{\tau}'}(\delta)$. Then we have $\kappa(\bar{\tau}) = \kappa(\bar{\tau}') - k$, $i_1 = i'_1$ and $\bar{c}_{\bar{\tau},j}(\delta) = \delta^k \cdot \bar{c}_{\bar{\tau}',j}(\delta)$ for any $j \in \mathbb{N}$.

Proof. Put $s := s_{\bar{\tau}}$. Viewing $v_{\bar{\tau}}(\delta)/(\delta - 1)$ and $\delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1)$ as functions of δ (see (17)), we expand them into Taylor series around infinity:

$$\frac{v_{\bar{\tau}}(\delta)}{\delta - 1} = -s_{i_1} \cdot \delta^{-\varepsilon_1(\bar{\tau})} - s_{i_2} \cdot \delta^{-\varepsilon_2(\bar{\tau})} - \dots - s_{i_{|\bar{\tau}|}} \cdot \delta^{-\varepsilon_{|\bar{\tau}|}(\bar{\tau})} - s_{i_{|\bar{\tau}|+1}} \cdot \delta^{-\varepsilon_{|\bar{\tau}|+1}(\bar{\tau})} - \dots, \quad (39)$$

$$\frac{\delta^k \cdot v_{\bar{\tau}'}(\delta)}{\delta - 1} = -s_{i'_1} \cdot \delta^{-\varepsilon_1(\bar{\tau}') + k} - \dots - s_{i'_{|\bar{\tau}'|}} \cdot \delta^{-\varepsilon_{|\bar{\tau}'|}(\bar{\tau}') + k} - s_{i'_{|\bar{\tau}'|+1}} \cdot \delta^{-\varepsilon_{|\bar{\tau}'|+1}(\bar{\tau}') + k} - \dots. \quad (40)$$

In view of these expressions, the coefficient of δ^{-l} is either $-s_{\bullet}$ or 0. Now assume the contrary that $v_{\bar{\tau}}(\delta)/(\delta - 1)$ and $\delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1)$ have different coefficients. Let l_1 and l_2 be the minimal integers such that $v_{\bar{\tau}}(\delta)/(\delta - 1)$ and $\delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1)$ have the coefficients $-s_{m_1}$ and $-s_{m_2}$ of δ^{-l_1} and of δ^{-l_2} which are different from the coefficient of δ^{-l_1} in $\delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1)$ and the coefficient of δ^{-l_2} in $v_{\bar{\tau}}(\delta)/(\delta - 1)$ for some $1 \leq m_1 \leq n$ and $1 \leq m_2 \leq n$ respectively. Note that $s_1 > s_2 > \dots > s_n$ and $\varepsilon_{m+1}(\bar{\tau}) - \varepsilon_m(\bar{\tau}) \geq 3$ for any $\bar{\tau}$ and m . Thus, $v_{\bar{\tau}}(\delta)/(\delta - 1) - \delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1) = 0$ satisfies the estimates

$$s_{m_1} \delta^{-l_1} - s_{m_2} \delta^{-l_2} - s_1 \frac{\delta^{-l_2}}{\delta^3 - 1} < \frac{v_{\bar{\tau}}(\delta)}{\delta - 1} - \frac{\delta^k \cdot v_{\bar{\tau}'}(\delta)}{\delta - 1} < s_{m_1} \delta^{-l_1} - s_{m_2} \delta^{-l_2} + s_1 \frac{\delta^{-l_1}}{\delta^3 - 1}.$$

If $l_1 > l_2$, then it follows that

$$0 < s_{m_1} \delta^{-l_1} - s_{m_2} \delta^{-l_2} + s_1 \frac{\delta^{-l_1}}{\delta^3 - 1} < s_1 \delta^{-l_1} - s_1 z_1(n)^{n-1} \delta^{-l_1+1} + s_1 \frac{\delta^{-l_1}}{\delta^3 - 1} < s_1 \delta^{-l_1} g_1(n),$$

which contradicts Lemma 7.4. On the other hand, if $l_1 = l_2$ and $m_1 > m_2$, then we have

$$0 < s_{m_1} \delta^{-l_1} - s_{m_2} \delta^{-l_1} + s_1 \frac{\delta^{-l_1}}{\delta^3 - 1} < s_1 z_2(n)^{m_1-1} \delta^{-l_1} - s_1 z_1(n)^{m_1-2} \delta^{-l_1} + s_1 \frac{\delta^{-l_1}}{\delta^3 - 1} < -s_1 \delta^{-l_1} g_2(n),$$

where the last inequality is a consequence of the fact that $z_2(n)^{m_1-1} - z_1(n)^{m_1-2} = -z_2(n)^{m_1-2} \left(\left(\frac{z_1(n)}{z_2(n)} \right)^{m_1-2} - z_2(n) \right)$ is monotone increasing with respect to m_1 since $0 < z_2(n), \frac{z_1(n)}{z_2(n)} < 1$ and $\left(\frac{z_1(n)}{z_2(n)} \right)^{m_1-2} - z_2(n) > \frac{g_2(n)}{z_2(n)^{m_1-2}} > 0$. This contradicts Lemma 7.4. In a similar manner, if $l_1 < l_2$, then it follows that

$$0 > s_{m_1} \delta^{-l_1} - s_{m_2} \delta^{-l_2} - s_1 \frac{\delta^{-l_2}}{\delta^3 - 1} > s_1 z_1(n)^{n-1} \delta^{-l_2+1} - s_1 \delta^{-l_2} - s_1 \frac{\delta^{-l_2}}{\delta^3 - 1} > -s_1 \delta^{-l_2} g_1(n),$$

which is a contradiction. On the other hand, if $l_1 = l_2$ and $m_1 < m_2$, then we have

$$0 > s_{m_1} \delta^{-l_2} - s_{m_2} \delta^{-l_2} - s_1 \frac{\delta^{-l_2}}{\delta^3 - 1} > s_1 z_1(n)^{m_2-2} \delta^{-l_2} - s_1 z_2(n)^{m_2-1} \delta^{-l_2} - s_1 \frac{\delta^{-l_2}}{\delta^3 - 1} > s_1 \delta^{-l_2} g_2(n),$$

which is a contradiction. Thus, $v_{\bar{\tau}}(\delta)/(\delta - 1)$ and $\delta^k \cdot v_{\bar{\tau}'}(\delta)/(\delta - 1)$ have the same coefficients. In particular, we have $i_1 = i'_1$ and $\kappa(\bar{\tau}) = \varepsilon_1(\bar{\tau}) = \varepsilon_1(\bar{\tau}') - k = \kappa(\bar{\tau}') - k$. Moreover, $\bar{c}_{\bar{\tau},j}(\delta) = \delta^k \cdot \bar{c}_{\bar{\tau}',j}(\delta)$ holds since $\bar{c}_{\bar{\tau},j}(\delta)/(\delta - 1)$ and $\delta^k \cdot \bar{c}_{\bar{\tau}',j}(\delta)/(\delta - 1)$ are the sums of the terms δ^l in (39) and (40) whose coefficients are equal to s_j , respectively. Therefore, the lemma is established. \square

Recall that if $\bar{c}_{\bar{\tau},j}(d) \neq 0$, then it may be expressed as

$$\bar{c}_{\bar{\tau},j}(d) = -\frac{(d-1) \cdot (d^{\eta_j} + \epsilon_{j,1} d^{\eta_{j,1}} + \epsilon_{j,2} d^{\eta_{j,2}})}{d^{\varepsilon_{|\bar{\tau}|}} - 1}$$

for some $\eta_j := \eta_j(\bar{\tau}) < \eta_{j,1} < \eta_{j,2}$ and $\epsilon_{j,k} := \epsilon_{j,k}(\bar{\tau}) \in \{0, 1\}$. In view of this expression, one has

$$\eta_i(\bar{\tau}) = 0, \quad \eta_{i_{|\bar{\tau}|-1}}(\bar{\tau}) = \kappa(\bar{\tau}_{|\bar{\tau}|-1}), \quad \eta_j(\bar{\tau}) > 0 \quad (j \neq i).$$

Lemma 7.6 For a given $d > 2$, we put

$$\mathcal{F}_1(m_1; k) := \frac{d^{m_1}}{d^k - 1}, \quad \mathcal{F}_2(m_1, m_2; k) := \frac{d^{m_1} + d^{m_2}}{d^k - 1}, \quad \mathcal{F}_3(m_1, m_2, m_3; k) := \frac{d^{m_1} + d^{m_2} + d^{m_3}}{d^k - 1},$$

where $m_1 < m_2 < m_3 \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. Then,

- (1) if $\mathcal{F}_j(m_{1,1}, \dots, m_{1,j}; k_1) = \mathcal{F}_j(m_{2,1}, \dots, m_{2,j}; k_2)$ for $j = 1, 2, 3$, then we have $(m_{1,1}, \dots, m_{1,j}, k_1) = (m_{2,1}, \dots, m_{2,j}, k_2)$,
- (2) if $\mathcal{F}_2(m_{1,1}, m_{1,2}; k_1) = \mathcal{F}_1(m_{2,1}; k_2)$, then we have $(m_{1,1}, m_{1,2}, k_1) = (m, m + k, 2k)$ and $(m_{2,1}, k_2) = (m, k)$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$,
- (3) if $\mathcal{F}_3(m_{1,1}, m_{1,2}, m_{1,3}; k_1) = \mathcal{F}_1(m_{2,1}; k_2)$, then we have $(m_{1,1}, m_{1,2}, m_{1,3}, k_1) = (m, m + k, m + 2k, 3k)$ and $(m_{2,1}, k_2) = (m, k)$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$,
- (4) if $\mathcal{F}_3(m_{1,1}, m_{1,2}, m_{1,3}; k_1) = \mathcal{F}_2(m_{2,1}, m_{2,2}; k_2)$, then we have $(m_{1,1}, m_{1,2}, m_{1,3}, k_1) = (m, m + k, m + 2k, 3k)$ and $(m_{2,1}, m_{2,2}, k_2) = (m, m + k, 2k)$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.

In particular, if $\mathcal{F}_{j_1}(m_{1,1}, \dots, m_{1,j_1}; k_1) = \mathcal{F}_{j_2}(m_{2,1}, \dots, m_{2,j_2}; k_2)$ for some $j_1, j_2 = 1, 2, 3$, then we have $m_{1,1} = m_{2,1}$.

Proof. We only discuss the case $\mathcal{F}_1(m_{1,1}; k_1) = \mathcal{F}_1(m_{2,1}; k_2)$ as the remaining cases can be treated in a similar manner. Moreover, multiplying the both sides by $d^{-m_{1,1}}$, we may assume the relation $\mathcal{F}_1(0; k_1) = \mathcal{F}_1(m_{2,1}; k_2)$, which yields

$$d^{k_1+m_{2,1}} + 1 = d^{k_2} + d^{m_{2,1}}.$$

If $k_1 + m_{2,1} > k_2$, then one has $d^{k_1+m_{2,1}} + 1 > d^{k_2} + d^{m_{2,1}}$ from the assumption that $d > 2$. Similarly, if $k_1 + m_{2,1} < k_2$, then one has $d^{k_1+m_{2,1}} + 1 < d^{k_2} + d^{m_{2,1}}$. This means that $k_1 + m_{2,1} = k_2$, and thus $m_{2,1} = 0$ since $d^{m_{2,1}} = 1$. Therefore, we have $(0, k_1) = (m_{2,1}, k_2)$ and establish the lemma. \square

Lemma 7.7 Assume that $\bar{c}_{\bar{t},j}(d) = d^k \cdot \bar{c}_{\bar{t}',j}(d) \neq 0$ for some $j \in \{1, \dots, n\}$ and $d > 2$. Then we have $\eta_j(\bar{t}) = k + \eta_j(\bar{t}')$.

Proof. Using the notation of Lemma 7.6, the relation $\bar{c}_{\bar{t},j}(d) = d^k \cdot \bar{c}_{\bar{t}',j}(d)$ yields $\mathcal{F}_{j_1}(\eta_j(\bar{t}), \dots) = \mathcal{F}_{j_2}(k + \eta_j(\bar{t}'), \dots)$ for some $j_1, j_2 \in \{1, 2, 3\}$, and thus $\eta_j(\bar{t}) = k + \eta_j(\bar{t}')$, which establishes the lemma. \square

Corollary 7.8 Assume that $v_{\bar{t}}(\delta) = \delta^k \cdot v_{\bar{t}'}(\delta)$. Then we have $k = 0$ and $i = i'$.

Proof. By Lemma 7.5, one has $\bar{c}_{\bar{t},j}(\delta) = \delta^k \cdot \bar{c}_{\bar{t}',j}(\delta)$ for any $j \in \{1, \dots, n\}$. In particular, it follows that $\bar{c}_{\bar{t},i'}(\delta) = \delta^k \cdot \bar{c}_{\bar{t}',i'}(\delta) \neq 0$ and $\bar{c}_{\bar{t},i}(\delta) = \delta^k \cdot \bar{c}_{\bar{t}',i}(\delta) \neq 0$. Moreover, since $\eta_{i'}(\bar{t}) = k + \eta_{i'}(\bar{t}') = k$, $0 = \eta_i(\bar{t}) = k + \eta_i(\bar{t}')$ by Lemma 7.7, and $\eta_{i'}(\bar{t}), \eta_i(\bar{t}') \geq 0$, we have $\eta_{i'}(\bar{t}) = \eta_i(\bar{t}') = 0$, which yields $i = i'$ and thus $k = 0$. \square

Corollary 7.9 Assume that $v_{\bar{t}}(\delta) = v_{\bar{t}'}(\delta)$. Then we have $v_{\bar{t}_m}(\delta) = v_{\bar{t}'_m}(\delta)$, $\kappa(\bar{t}_m) = \kappa(\bar{t}'_m)$ and $i_m = i'_m$ for any $m \geq 0$.

Proof. Let us prove the corollary by induction on m . For $m = 0$, the statement immediately follows from Lemma 7.5 and Corollary 7.8. Assume that the statement holds for some m . Then Corollary 7.8 and Lemma 7.5 show that $i_m = i'_m$ and $i_{m+1} = i'_{m+1}$. Moreover, since $u_{\bar{\tau}_{m+1}} = \delta^{\kappa(\bar{\tau}_m)-1} \cdot v_{\bar{\tau}_m} = \delta^{\kappa(\bar{\tau}'_m)-1} \cdot v_{\bar{\tau}'_m} = u_{\bar{\tau}'_{m+1}}$, we have $v_{\bar{\tau}_{m+1}} = \delta \cdot u_{\bar{\tau}_{m+1}} + s_{i_{m+1}} = \delta \cdot u_{\bar{\tau}'_{m+1}} + s_{i'_{m+1}} = v_{\bar{\tau}'_{m+1}}$, and thus $\kappa(\bar{\tau}_{m+1}) = \kappa(\bar{\tau}'_{m+1})$ by Lemma 7.5. Therefore, the statement is verified for $m + 1$ and the induction is complete. \square

Proof of Proposition 6.5. From Corollaries 7.8 and 7.9, if the relation $\delta^k \cdot v_{\bar{\tau}}(\delta) = v_{\bar{\tau}'}(\delta)$ holds, then one has $k = 0$, $i_m = i'_m$ and $\kappa(\bar{\tau}_m) = \kappa(\bar{\tau}'_m)$ for any $m \geq 0$. Conversely, it is easily seen that if $i_m = i'_m$ and $\kappa(\bar{\tau}_m) = \kappa(\bar{\tau}'_m)$ for any $m \geq 0$ then $v_{\bar{\tau}}(\delta) = v_{\bar{\tau}'}(\delta)$ holds. In particular, we have $\bar{\Gamma}_{\tau}^{(2)} \cap P(\tau) = \{\alpha_{\bar{\tau}, \bar{\tau}'}^0 \mid i_m = i'_m, \kappa(\bar{\tau}_m) = \kappa(\bar{\tau}'_m), m \geq 0\}$ from (37) and Lemma 3.1. Moreover, assume that τ satisfies condition (3) in Theorem 1.5. For $\alpha_{\bar{\tau}, \bar{\tau}'}^k \in \bar{\Gamma}_{\tau}^{(2)} \cap P(\tau)$, it follows that $\bar{\tau} \prec_i \bar{\tau}'$ if and only if $\bar{\tau}_1 \prec_{i_1} \bar{\tau}'_1$ for a fixed $(\prec_i) \in \mathcal{T}(\tau)$ (see also Definition 2.3). Since $\theta_{i, i'}(k) = 0$ and $\mu(\bar{\tau}) = \mu(\bar{\tau}')$, we have $\alpha_{\bar{\tau}, \bar{\tau}'}^k \notin \Gamma_{\tau}^{(2)}$ and $\Gamma_{\tau}^{(2)} \cap P(\tau) = \emptyset$, which establishes Proposition 6.5. \square

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